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ESTIMATION OF DOPPLER SHIFT AND DIFFERENTIAL DOPPLER SHIFT.(U)
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DEPARTMENT OF ENGINEERING
AND APPLIED SCIENCE

YALE UNIVERSITY

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ESTIMATION OF DOPPLER SHIFT

AND DIFFERENTIAL DOPPLER SHIFT.

by

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Peter M./Schultheiss

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Technical Report.

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Abstract

This report is concerned with the measurement of frequency (Doppler shift) by a collection of sensors which may be grouped into two or more clusters (subarrays). The signal may be a sinusoid, or it may be a narrow-band Gaussian process whose center frequency is then the quantity of interest. For much, though not all, of the analysis, it is assumed that the noise is incoherent from sensor to sensor. The signal wavefront is assumed to be perfectly coherent over all sensors, but the angle subtended at the source by the various sensors may be sufficiently large so that differential Doppler shifts cannot be ignored. The following are some of the more general conclusions.

I. No differential Doppler shift between sensors

1. Sinusoidal signal
   a. The mean square frequency estimation error varies as T^{-3}, where T is the observation time. It varies as the inverse second power of the post-beamforming signal to noise ratio.
   b. The frequency estimate is not degraded by lack of prior knowledge of bearing and range (i.e. sensor to sensor delay of the signal). Frequency estimation and bearing-range estimation are therefore uncoupled.
   c. The mean square estimation error of sensor to sensor signal delay varies as T^{-1}. Hence the frequency estimate becomes extremely accurate long before the delay estimate.

2. Narrowband Gaussian signal, TW >> 1.

The frequency of interest is here the center frequency of the narrowband Gaussian signal. This particular computation has been carried out for completely arbitrary array geometry and noise field properties.

a. The mean square estimation error of the center frequency ω_0 varies as T^{-1}. Accuracy therefore improves at the same time rate as for delay estimation but much slower than for frequency estimation using a pure sinusoid.

b. The mean square estimation error of ω_0 depends on two quantities aside from T: the post-beamforming signal to noise ratio of the beamformer optimally matched to the noise field and the slope (on a db scale) of the signal spectrum.

c. The coupling between center frequency estimation and bearing estimation depends on the derivative of the array gain with respect to bearing in the neighborhood of the true bearing. If this derivative is zero there is no coupling. Thus bearing and frequency estimation are completely uncoupled for a spatially incoherent noise field. They are only weakly coupled in many, if not most, practically interesting situations. For strong coupling the noise field would have to exhibit a rapidly varying spatial structure near the signal bearing (e.g. an interference close to the signal in bearing).

3. Narrowband Gaussian signal, TW << 1.

When the time-bandwidth product is small compared to unity, a Gaussian signal looks essentially like a sinusoidal signal of unknown amplitude and phase. The amplitude estimate is not coupled to the frequency estimate so that the results of section (1) are applicable. One must

note, however, that the apparently very accurate frequency estimate (T^{-3} dependent) is a short term frequency estimate. To define a center frequency ω_0 of the narrowband signal one inherently requires an observation time large compared with the inverse signal bandwidth.

II. Differential Doppler estimation.

The differential Doppler estimation problem was studied for two or three sensor groups (subarrays) which were treated as equivalent single sensors.

1. Sinusoidal signal

   The mean square error for differential Doppler estimation varies as T^{-3}, just as that of the frequency estimate. For two subarrays it differs from the latter by 3db because there are two opportunities to measure frequency but only one to measure frequency difference. Subtraction of two separate frequency estimates to obtain the differential Doppler shift leads to the same error as direct estimation of the differential Doppler shift.

2. Narrowband Gaussian signal.

   Since narrowband signals with $TW \ll 1$ have the properties of sinusoids, only the case $TW \gg 1$ was considered. An analytically important assumption places an upper bound on values of T for which the results are valid: If the signal bandwidth is σ, the differential source velocity Δv and the velocity of sound c, T must be shorter than the period of a sinusoid of frequency $(\Delta v/c)\sigma$.

   a. The mean square estimation error of differential Doppler shift varies as T^{-3}, just as the frequency estimate of a sinusoid. The

estimation process for differential Doppler shift is coherent whereas that for center frequency is incoherent.

b. When the signal to noise ratio in the signal band is high, the differential Doppler estimate using narrowband signals has the same mean square error as an estimate using sinusoidal signals.

c. The only important difference between differential Doppler estimates using narrowband signals at high and low signal to noise ratios is the signal-to-noise (S/N) dependence of the mean square error. At high signal to noise ratios it varies as $(S/N)^{-1}$, at low signal to noise ratios as $(S/N)^{-2}$.

d. Because of the different T dependence, efforts to obtain differential Doppler shift from separate measurements of center frequency at each subarray lead to a much poorer estimate. The loss in performance is proportional to the square of the TW product.

e. The estimate of differential Doppler shift is completely uncoupled from the estimate of bearing, range and parameters describing spectral shape. Differential Doppler measurements can therefore be carried out without concern about most other parameters which are likely to be unknown in practice.

f. When there are three subarrays, there are two differential Doppler shifts. Their estimates are coupled, but neither is coupled to the estimates of bearing, range or spectral parameters. When the signal to noise ratio in the signal band is high, the estimation error for each differential Doppler shift is the same as that from the appropriate pair of subarrays. When the in-band signal to noise ratio is low, a small gain can be made by processing the three subarray outputs as a unit rather than obtaining separate differential Doppler shifts from two pairs of subarrays.

I. Introduction

This report summarizes work carried out under contract N66001-76-C-0182 between Yale University and the Naval Ocean Systems Center, San Diego. It describes the continuation of studies initiated under contract N66001-75-C-0210 and reported in June 1976. [1]

The overall objective of both contracts was to obtain information about an acoustic source by means of a large array of sensors which may be grouped into several subarrays. Reference [1] was concerned with source localization in bearing and range. Here we turn to the problem of Doppler estimation and its coupling to bearing and range estimation.

Much of the basic theory developed in [1] remains applicable to the present study and will therefore only be referenced briefly when needed. In particular, we shall retain most of the basic assumptions:

(a) The source radiates either a sinusoid or a narrowband Gaussian signal.

(b) The signal wavefront is perfectly coherent over the entire receiving array.

(c) The noise is Gaussian with a bandwidth broad compared with that of the signal. Much, though not all, of the analysis will also assume that the noise is spatially incoherent.

(d) The frequency or center frequency of the signal observed at any given sensor does not change significantly over the observation period.

However, in contrast to the earlier work, the signal frequency or center frequency need not be the same at each sensor. Thus we allow Doppler shifts and even differential Doppler shifts from sensor to sensor. Assumption (d) does not preclude source motion; it merely says that the

source-receiver geometry does not change significantly during the observation period.

A major conclusion of the previous report was that, for most purposes, each subarray could be treated as a single sensor whose relatively weak directional properties were of only secondary importance. Thus one could, without significant loss, beamform at each subarray and treat the resulting beams as the outputs of equivalent single sensors. The subarrays serve only to enhance the signal to noise ratio of these equivalent sensors. If the source is sufficiently distant to warrant the use of more than one subarray, there will be very little differential Doppler shift over the elements of any one subarray. Beamforming at the subarray level should therefore not destroy any significant amount of Doppler information. We shall find this to be true in the first, simple formulations of the Doppler estimation problem. Thereafter we shall then treat each subarray as a single equivalent sensor with appropriately enhanced signal to noise ratio.

As in Ref. [1] we shall use the Cramér-Rao inequality to set lower bounds on attainable estimation errors. In the limit of large observation times, this lower bound is approached by the error of a maximum likelihood estimator [2]. At least in this important limit, therefore, the calculated bound can be reached by an actual instrumentation.

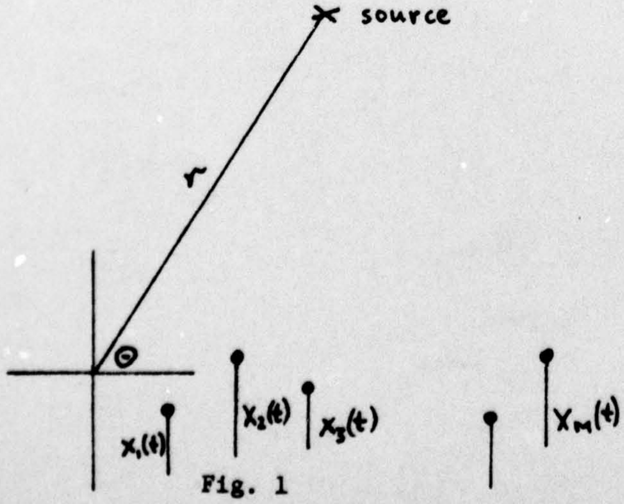

Fig. 1

In the most general setting of the problem we have a geometry such as shown in Figure 1.

A source located at range r and bearing Θ (measured in some convenient polar coordinate system) radiates a signal which is received

(together with the inevitable noise) at M sensors in known but perfectly arbitrary locations. The output of the i^{th} sensor is

$$x_i(t) = s_i(t) + n_i(t) \tag{1}$$

$s_i(t)$ is the signal component and $n_i(t)$ the noise component. We construct a data vector $\underline{x}$ as follows:

(1) Represent $x_i(t)$ by a set of n numbers [e.g. the time samples $x_i(t_1), x_i(t_2),\ldots,x_i(t_n)$ or the Fourier coefficients c_{1i}, $c_{2i},\ldots,c_{ni}$].

(2) Concatenate these n number sets for all index values i [e.g. $x_1(t_1),\ldots,x_1(t_n), x_2(t_1),\ldots,x_2(t_n),\ldots x_M(t_1),\ldots x_M(t_n)$].

The resultant data vector $\underline{x}$ therefore has dimension nM.

The Cramér-Rao technique requires computation of the likelihood function

$$\Lambda = p(\underline{x}/\alpha_1,\alpha_2,\ldots\alpha_r) \tag{2}$$

$\alpha_1, \alpha_2,\ldots,\alpha_r$ are the parameters to be estimated (e.g. Doppler shift, bearing, range, etc.). The right side of Eq. (2) is the conditional probability density for the data vector $\underline{x}$ when the unknown parameters have values $\alpha_1,\alpha_2,\ldots,\alpha_r$.

Next one must form the Fisher information matrix J whose elements $j_{k\ell}$ are given by

$$j_{k\ell} = -E\{\frac{\partial^2 \log \Lambda}{\partial\alpha_k\,\partial\alpha_\ell}\} \tag{3}$$

E{ } is the expectation of the bracketed quantity. The Cramér-Rao inequality asserts that the (unbiased) mean square estimation error $D^2(\hat{\alpha}_k)$ of the k^{th} parameter is bounded by the kk element of the inverse matrix J^{-1}.

$$D^2(\hat{\alpha}_k) \geq [J^{-1}]_{kk} \tag{4}$$

Off-diagnol elements of J^{-1} measure the coupling between estimation errors.

II. Frequency Estimation - No Differential Doppler Shift

We begin with the problem of frequency estimation using a single subarray. The elements of such an array are presumably spaced rather tightly and differential Doppler shifts from sensor to sensor are negligible for all but very nearby sources. The frequency or center frequency of the signal component received by each sensor is therefore the same. Our objective is to estimate that frequency or center frequency, possible in the presence of other initially unknown parameters such as source bearing or range.[1]

1. Sinusoidal signal

We begin with the simplest case, a source radiating a pure sinusoid. The Doppler shifted signal component at the receiving array is a sinusoid of frequency ω_o. Thus the output of the i^{th} sensor [Fig. 1] is

$$x_i(t) = A_i \sin[\omega_o(t-\tau_i) - \phi] + n_i(t) \tag{5}$$

τ_i is the signal travel time from the source to the i^{th} sensor (bearing and range dependent) and $n_i(t)$ is the noise component at the i^{th} sensor. For the present computation we shall assume that the noise is Gaussian, spatially incoherent (statistically independent from sensor to sensor) and spectrally white over a band of W Hertz beginning at zero frequency.

Calculation of the Fisher information matrix proceeds much as in Ref. [1]. The data vector consists of time samples of the sensor outputs taken at the Nyquist rate (i.e., at intervals of 1/2W seconds). Because the noise is both spatially and spectrally white as well as Gaussian,

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[1]Note that this formulation of the problem does not permit separate estimation of radiated source frequency and Doppler shift. Only the sum of the two frequencies is accessible to measurement and hence to estimation.

all of the samples are statistically independent and the likelihood function assumes the simple form

$$\Lambda = C \exp -\{\sum_{i=1}^{M} \sum_{j=1}^{n} \frac{1}{2N_iW}[x_i(t_j) - s_i(t_j)]^2\} \tag{6}$$

N_i is the spectral level of the noise received at the i^{th} sensor and N_iW is the average noise power of each sample at that sensor. C is a normalizing constant which insures that Λ is a probability density. Note that j serves as an index for the time samples, i for the sensor locations (space samples).

As a typical element of the Fisher information matrix consider $-E\{\partial^2 \log \Lambda/\partial\omega_o^2\}$. Differentiating the natural logarithm of Eq. (6) twice with respect to ω_o one obtains:

$$\frac{\partial^2 \log \Lambda}{\partial\omega_o^2} = \sum_{i=1}^{M} \sum_{j=1}^{n} \frac{1}{N_iW}\{-[\frac{\partial s_i(t_j)}{\partial\omega_o}]^2 + [x_i(t_j) - s_i(t_j)]\frac{\partial^2 s_i(t_j)}{\partial\omega_o^2}\} \tag{7}$$

$x_i(t_j) - s_i(t_j)$ is simply the j^{th} sample of the noise at the i^{th} sensor. Since the noise has zero mean, the averaging operation eliminates the second term of Eq. (7) and one is left with

$$-E\{\frac{\partial^2 \log \Lambda}{\partial\omega_o^2}\} = \sum_{i=1}^{M} \sum_{j=1}^{n} \frac{1}{N_iW}[\frac{\partial s_i(t_j)}{\partial\omega_o}]^2 \tag{8}$$

From Eq. (5)

$$s_i(t_j) = A_i \sin[\omega_o(t_j - \tau_i) - \phi] \tag{9}$$

Hence

$$-E\{\frac{\partial^2 \log \Lambda}{\partial\omega_o^2}\} = \sum_{i=1}^{M} \sum_{j=1}^{n} \frac{A_i^2}{N_iW}(t_j-\tau_i)^2 \cos^2[\omega_o(t_j-\tau_i) - \phi] \tag{10}$$

When the noise bandwidth W is sufficiently large so that there are many samples in a signal period one can approximate the j sum in Eq. (10) by an integral.

$$-E\{\frac{\partial^2 \log \Lambda}{\partial \omega_o^2}\} = 2W \sum_{i=1}^{M} \sum_{j} \frac{A_i^2}{N_i W}(t_j - \tau_i)^2 \cos^2[\omega_o(t_j - \tau_i) - \phi] \quad \Delta t$$

$$\rightarrow 2 \sum_{i=1}^{M} \frac{A_i^2}{N_i} \int_{t_o}^{t_o+T} (t-\tau_i)^2 \cos^2[\omega_o(t-\tau_i) - \phi]dt \tag{11}$$

In the first version of this equation $\Delta t = 1/2W$ is the time increment from sample to sample. (t_o, t_o+T) is the observation interval.

In practice T is almost certainly very large compared with the signal period. In that case double frequency terms make negligible contributions to the integral and one obtains

$$-E\{\frac{\partial^2 \log \Lambda}{\partial \omega_o^2}\} = \frac{1}{3} \sum_{i=1}^{M} \frac{A_i^2}{N_i} \{(T+t_o - \tau_i)^3 - (t_o-\tau_i)^3\} \tag{12}$$

Suppose the signal travel time from the source to the center of the array is τ_o. One can absorb that quantity into t_o with the definition $t_1 = t_o - \tau_o$. The differential delays $\tau_i - \tau_o$ are almost certainly small compared with T and one can then write to an excellent approximation

$$-E\{\frac{\partial^2 \log \Lambda}{\partial \omega_o^2}\} = \frac{1}{3} \sum_{i=1}^{M} \frac{A_i^2}{N_i}[(t_1+T)^3 - t_1^3] \tag{13}$$

If ω_o is the only unknown parameter, Eq. (13) is the only element in the Fisher information matrix and its reciprocal should be the minimum mean square estimation error. This is a correct but highly misleading statement. Eq. (13) depends strongly on the largely arbitrary t_1. One can apparently reduce the estimation error drastically by merely starting the observations later! There is a simple explanation for this paradox: We are making the completely unrealistic assumption that the phase angle ϕ is known. If this were actually true we could use the phase shift $\omega_o t_o$ accumulated prior to the start of the observation interval to improve

the estimate of ω_o. In practice, this is not an option open to us. The minimal problem which has physical meaning is therefore that of jointly estimating ω_o and ϕ (even though the latter is ultimately without interest).

Computation of the remaining elements of the Fisher information matrix for ω_o and ϕ proceeds exactly as in Eqs. (7)-(13). The result is

$$J = \left[\begin{array}{c|c} \frac{1}{3}\sum_{i=1}^{M}\frac{A_i^2}{N_i}[(t_1+T)^3 - t_1^3] & -\frac{1}{2}\sum_{i=1}^{M}\frac{A_i^2}{N_i}[(t_1+T)^2 - t_1^2] \\ \hline -\frac{1}{2}\sum_{i=1}^{M}\frac{A_i^2}{N}[(t_1+T)^2 - t_1^2] & T\sum_{i=1}^{M}\frac{A_i^2}{N_i} \end{array}\right] \tag{14}$$

The determinant of J is easily computed

$$\text{Det } J = \frac{T^4}{12}\sum_{i=1}^{M}\sum_{k=1}^{M}\frac{A_i^2}{N_i}\frac{A_k^2}{N_k} \tag{15}$$

Hence the minimum mean square error for frequency estimation is

$$D^2(\hat{\omega}_o) = [J^{-1}]_{11} = \frac{12}{T^3\sum_{i=1}^{M}\frac{A_i^2}{N_i}} \tag{16}$$

This quantity is independentof t_1, as one would expect.

If the signal to noise ratio at each sensor is the same

$$\frac{A_i^2}{N_i} = \frac{A^2}{N_1}$$

Then

$$D^2(\hat{\omega}_o) = \frac{12}{T^3 M A^2/N_1} \tag{17}$$

$M A^2/N_1$ measures the output signal to noise ratio of a conventional beamformer. A single sensor with that signal to noise ratio would therefore be entirely equivalent to the array for frequency measuring purposes.

Next we must look into the coupling between frequency estimation on the one hand and bearing and range estimation on the other. Since bearing and range estimates are obtained from the sensor to sensor delay of the signal, we can gain the required insight by working with only two sensors and estimating the vector $(\omega_o, \Delta\tau, \phi)$, where $\Delta\tau$ is the signal delay between the two sensors.

The required Fisher information matrix is readily computed

$$J = \left[\begin{array}{c:c:c} \frac{2}{3}\frac{A^2}{N_1}(t_1+T)^3 - t_1^3] & -\frac{1}{2}\omega_o\frac{A^2}{N_1}[(t_1+T)^2 - t_1^2] & -\frac{A^2}{N_1}[(t_1+T)^2 - t_1^2] \\ \hdashline -\frac{1}{2}\omega_o\frac{A^2}{N_1}[(t_1+T)^2 - t_1^2] & \omega_o^2\, T\frac{A^2}{N_1} & \omega_o\, T\frac{A^2}{N_1} \\ \hdashline -\frac{A^2}{N_1}[(t_1+T)^2 - t_1^2] & \omega_o\, T\frac{A^2}{N_1} & 2\, T\frac{A^2}{N_1} \end{array}\right] \tag{18}$$

The matrix inversion is straightforward and one obtains the mean square estimation errors

$$D^2(\hat{\omega}_o) = \frac{6}{T^3\, A^2/N_1} \tag{19}$$

$$D^2(\Delta\hat{\tau}) = \frac{2}{\frac{A^2}{N_1}\,\omega_o^2\, T} \qquad (20)[1]$$

Note that the frequency estimation error improves with the cube of the observation time, whereas the delay estimation error only improves with the first power of the observation time. Perhaps the most informative comparison is that between the fractional errors in the two estimates

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[1] This ignores the ambiguity problem associated with sinusoidal signals. See Ref. [1].

$$\frac{D^2(\hat{\omega}_o)/\omega_o^{\;2}}{D^2(\hat{\Delta\tau})/(\Delta\tau\;)^2} = 3(\frac{\Delta\tau}{T})^2 \tag{21}$$

In practice T is likely to be many times larger than $\Delta\tau$. The frequency estimate therefore becomes exceedingly accurate before a reasonable delay estimate can be made.

Another interesting comparison is that between Eqs. (19) and (17). Since $M = 2$ in Eq. (19) the two are, in fact, identical. The frequency estimate is not degraded at all by lack of a priori knowledge of the relative delay (i.e., knowledge of source location). One would expect this relationship to be reciprocal and indeed one finds for known ω_o [using the last two rows and columns of Eq. (18)]

$$D^2(\Delta\tau)\Big|_{\omega_o \text{ known}} = \frac{2}{\frac{A^2}{N_1}\;\omega_o^{\;2}\;T} \tag{22}$$

This is identical with Eq. (20), confirming the independence of the frequency and delay estimates.

2. Narrowband Gaussian Signal. TW >> 1.

In many cases the signal observed at the receiving array is not a pure sinusoid. There is experimental evidence that it can often be modelled quite effectively as a narrowband Gaussian random process. The unknown parameter ω_o is now the center frequency of the power spectrum associated with this random process.

For the moment we consider observation times T large compared with the inverse bandwidth of the process (TW >> 1). If the bandwidth of the noise

is at least equally large, the most convenient form of the data vector uses Fourier coefficients. For TW >> 1 the Fourier coefficients associated with different frequencies are uncorrelated [3]. Thus the covariance matrix K of the data vector contains many zero entries, a feature which can be exploited by arranging the data vector as follows

$$\underline{x} = [c_{11}\,{}_{12},\ldots c_{1M},\ c_{21},c_{22},\ldots,c_{2M},\ldots,c_{n1},c_{n2},\ldots c_{nM}]^T \tag{23}$$

c_{ji} is the Fourier coefficient of frequency ω_j, measured at the i^{th} sensor. The covariance matrix K now assumes the block diagonal form

$$K = \begin{bmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_n \end{bmatrix} \tag{24}$$

K_j is the spatial covariance matrix of Fourier coefficients at frequency ω_j. Its elements are of the form $E\{c_{j\ell}c^*_{jk}\}$.

The basic theory of parameter estimation from a Gaussian data vector was developed in Ref. [4]. The general result, for unknown parameter α, is

$$D^2(\hat{\alpha}) = \{Tr[(K^{-1}\frac{dK}{d\alpha})^2]\}^{-1} \tag{25}$$

Tr[ ] stands for the trace of the bracketed matrix. It is evident from Eq. (24) that $(K^{-1}\frac{dK}{d\alpha})^2$ is block diagonal and its trace is therefore simply the sum of the traces of the separate blocks.

$$D^2(\hat{\alpha}) = \{\sum_{j=1}^{n} Tr[(K_j^{-1}\frac{dK_j}{d\alpha})^2]\}^{-1} \tag{26}$$

K_j can also be simplified a great deal. The signal and noise components of $\underline{x}$ are statistically independent. For an array of moderate dimensions the same signal and noise spectra are received at each sensor. In such a case one can write

$$K_j = S(\omega_j)P(\omega_j) + N(\omega_j)Q(\omega_j) \tag{27}$$

$S(\omega)$ and $N(\omega)$ are respectively the signal and noise spectra received at any one sensor. $P(\omega)$ is the spatial covariance matrix of the signal at frequency ω, normalized so that the diagonal elements are unity. $Q(\omega)$ is the similarly normalized spatial covariance matrix of the noise. The parameter ω_o, the center frequency of the signal spectrum, is contained only in the scalar function S( ). Eq. (26) therefore assumes the rather simple form

$$D^2(\hat{\omega}_o) = \{ \sum_{j=1}^{n} Tr([S(\omega_j)P(\omega_j) + N(\omega_j)Q(\omega_j)]^{-1} \frac{dS(\omega_j)}{d\omega_o} P(\omega_j))^2 \}^{-1} \tag{28}$$

If all of the spectral functions are essentially constant over frequency intervals of the order $\Delta\omega = 2\pi/T$ [this is essentially the large TW assumption] the j sum can be converted into an integral

$$D^2(\hat{\omega}_o) = \frac{2\pi}{T} \{ \int_0^\infty Tr([S(\omega)P(\omega) + N(\omega)Q(\omega)]^{-1} \frac{dS(\omega)}{d\omega_o} P(\omega))^2 d\omega \}^{-1} \tag{29}$$

To proceed further we assume that the signal wavefront is coherent over the entire array. The signal received at one sensor is simply a time shifted version of the one received at any other sensor. Under these circumstances the P matrix has rank 1 and can be written as an outer product of vectors

$$P = \underline{V}(\omega) \underline{V}^*(\omega) \tag{30}$$

$\underline{V}(\omega)$ is the "steering vector", the vector of relative delay of the signal at different sensors. $\underline{V}^*(\omega)$ is its transpose-conjugate. From a standard matrix identity

$$[S(\omega)\underline{V}(\omega)\underline{V}^*(\omega) + N(\omega)Q(\omega)]^{-1}$$

$$= \frac{1}{N(\omega)} Q^{-1}(\omega) - \frac{S(\omega)}{N^2(\omega)} \frac{Q^{-1}(\omega)\underline{V}(\omega)\underline{V}^*(\omega)Q^{-1}(\omega)}{1 + \frac{S(\omega)}{N(\omega)}G(\omega)} \tag{31}$$

The scalar

$$G(\omega) \equiv \underline{V}^*(\omega)Q^{-1}(\omega)\underline{V}(\omega) \tag{32}$$

is recognized as the array gain of the beamformer optimally matched to the given noise field.

Substituting Eq. (31) into Eq. (29) and temporarily dropping the argument ω for greater simplicity of notation one obtains

$$D^2(\hat{\omega}_o) = \frac{2\pi}{T}\{\int_o^\infty \mathrm{Tr}[\frac{1}{N}\frac{dS}{d\omega_o} Q^{-1}P - \frac{S}{N^2}\frac{dS}{d\omega_o}\frac{Q^{-1}\underline{V}\overbrace{\underline{V}^*Q^{-1}\underline{V}}^{G}\underline{V}^*}{1 + \frac{S}{N}G}]^2 d\omega\}^{-1}$$

$$= \frac{2\pi}{T}\{\int_o^\infty [\frac{1}{N}\frac{dS}{d\omega_o}(1 - \frac{\frac{S}{N}G}{1+\frac{S}{N}G})]^2 \mathrm{Tr}[(Q^{-1}P)^2]d\omega\}^{-1} \tag{33}$$

Furthermore

$$\mathrm{Tr}[(Q^{-1}P)^2] = \mathrm{Tr}[Q^{-1}\underline{V}\,\overbrace{\underline{V}^*Q^{-1}\underline{V}}^{G}\,\underline{V}^*]$$

$$= G\ \mathrm{Tr}[Q^{-1}\underline{V}\ \underline{V}^*] = G\ \mathrm{Tr}(\underline{V}^*Q^{-1}\underline{V}) = G^2 \tag{34}$$

It follows that

$$D^2(\hat{\omega}_o) = \frac{2\pi}{T}\{\int_o^\infty (\frac{\frac{S(\omega)}{N(\omega)}G(\omega)}{1 + \frac{S(\omega)}{N(\omega)}G(\omega)} \frac{\frac{dS(\omega)}{d\omega_o}}{S(\omega)})^2 d\omega\}^{-1} \tag{35}$$

Note that the spectral properties of signal and noise are quite general, as is the spatial structure of the noise.

Eq. (35) says several interesting things about the estimation of ω_o:

(1) The estimation error varies as T^{-1}. The comparable estimation error for a sinusoidal signal varied as T^{-3}. For the narrowband signals it is no longer true that frequency can be estimated much more rapidly than differential delay (bearing).

(2) Array geometry and spatial structure of the noise field enter Eq. (35) only via the scalar $G(\omega)$, the optimal array gain. $\frac{S(\omega)}{N(\omega)} G(\omega)$ is the output signal to noise ratio of the optimal beamformer. One can therefore beamform prior to frequency estimation without loss in performance as long as one uses the beam-former structure familiar from the optimal detection problem. Alternatively one can think in terms of processing the output of a single "equivalent" sensor whose signal to noise ratio is $\frac{S(\omega)}{N(\omega)}G(\omega)$.

(3) Aside from the output signal to noise ratio, Eq. (35) *depends only on*

$$\frac{\frac{dS(\omega)}{d\omega_o}}{S(\omega)} = \frac{d}{d\omega_o}[\log S(\omega)] \tag{36}$$

The critical factor is therefore the slope of the signal spectrum plotted on a logarithmic (db) scale.

Example

To gain some insight into the behavior of the frequency estimator characterized by Eq. (35), consider a particular example. Suppose the noise is incoherent from sensor to sensor $[G(\omega) = M]$ and spectrally white at each sensor $[N(\omega) = N_o]$. Suppose further that the signal has the spectrum

$$S(\omega) = S_o \exp\{-\frac{(\omega - \omega_o)^2}{2\sigma^2}\} \tag{37}$$

Then Eq. (35) becomes

$$D^2(\hat{\omega}_o) = \frac{2\pi}{T}\{\int_o^\infty \left(\frac{M\frac{S_o}{N_o}\exp[-\frac{(\omega-\omega_o)^2}{2\sigma^2}]}{1+M\frac{S_o}{N_o}\exp[-\frac{(\omega-\omega_o)^2}{2\sigma^2}]}\;\frac{\omega-\omega_o}{\sigma^2}\right)^2 d\omega\}^{-1} \tag{38}$$

If $M\frac{S_o}{N_o} << 1$ (i.e. the post-beamforming signal to noise ratio nowhere reaches unity) the integral is easily evaluated by ignoring the second term in the denominator. Thus for low signal to noise ratio

$$D^2(\hat{\omega}_o) = \frac{\sqrt{\pi}\,\sigma}{2T(M\frac{S_o}{N_o})^2} \tag{39}$$

If $M\frac{S_o}{N_o} >> 1$ (high signal to noise ratio in the signal band) an appropriate approximation is

$$\frac{M\frac{S_o}{N_o}\exp[-\frac{(\omega-\omega_o)^2}{2\sigma^2}]}{1+M\frac{S_o}{N_o}\exp[-\frac{(\omega-\omega_o)^2}{2\sigma^2}]} \cong \begin{cases} 1 \text{ for } |\omega-\omega_o| \leq \Delta\omega \\ M\frac{S_o}{N_o}\exp[-\frac{(\omega-\omega_o)^2}{2\sigma^2}] \text{ for } |\omega-\omega_o| > \Delta\omega \end{cases} \tag{40}$$

$\Delta\omega$ is the value of $|\omega-\omega_o|$ at which the second denominator term reaches unity, i.e.,

$$\Delta\omega = \sqrt{2}\,\sigma\sqrt{\log(M\frac{S_o}{N_o})}^{\,1} \tag{41}$$

The integration of Eq. (38) can now be performed. It turns out that the range $|\omega-\omega_o| > \Delta\omega$ contributes very little to the result and one obtains

$$D^2(\hat{\omega}_o) \cong \frac{3\pi}{2\sqrt{2}\,T}\;\frac{\sigma}{[\log(M\frac{S_o}{N_o})]^{3/2}} \tag{42}$$

The principal difference between Eqs. (39) and (42) is the signal-to-noise dependence. When the post-beamforming signal to noise ratio is low the

---

[1] log( ) is the logarithm to base e.

the mean square error varies as the inverse square of that signal to noise ratio. For large post-beamforming signal to noise ratios the dependence becomes quite weak. We note that $2\Delta\omega$, the quantity which determines that behavior, is simply the frequency band over which the spectral level of the signal exceeds that of the noise. Its value is therefore quite critically dependent on the shape of the signal spectrum and no general significance should be attached to the logarithmic form appearing in Eq. (42).

Simultaneous estimation of frequency and bearing.

We must now discuss the interdependence between the center frequency estimate and the location estimate. Since we are not considering differential Doppler shifts, we are presumably dealing with a single array whose dimensions are small compared with the distance to the source. In that case the array furnishes little useful information concerning range but the differential delays yield a very direct measure of bearing. We therefore pose the problem of jointly estimating center frequency ω_o and bearing θ.

The basic theory of joint estimation from a Gaussian data vector is worked out in Ref. [4]. For the two parameter case one finds

$$D^2(\hat{\omega}_o) = \frac{D^2(\hat{\omega}_o)\Big|_{\theta \text{ known}}}{1-\rho} \tag{43}$$

where

$$\rho = \frac{\left[\mathrm{Tr}\left(K^{-1}\frac{\partial K}{\partial \omega_o}K^{-1}\frac{\partial K}{\partial \theta}\right)\right]^2}{\mathrm{Tr}\left[\left(K^{-1}\frac{\partial K}{\partial \omega_o}\right)^2\right]\,\mathrm{Tr}\left[\left(K^{-1}\frac{\partial K}{\partial \theta}\right)^2\right]} \tag{44}$$

ρ measures the coupling between the two estimates and $(1 - \rho)^{-1}$ is the degradation in the ω_o estimate due to lack of prior knowledge of θ.

The first term in the denominator of Eq. (44) is simply $[D^2(\hat{\omega}_o)\big|_{\theta \text{ known}}]^{-1}$ and is therefore available immediately from Eq. (35). Similarly, the second term in the denominator is $[D^2(\hat{\theta})\big|_{\omega_o \text{ known}}]^{-1}$. It is derived in Ref. [4] (p. 96) and is simply restated here for reference.

$$\mathrm{Tr}[(K^{-1} \frac{\partial K}{\partial \theta})^2] =$$

$$\frac{T}{2\pi} \int_0^\infty \frac{(\frac{S(\omega)}{N(\omega)})^2}{1 + \frac{S(\omega)}{N(\omega)} G(\omega)} [\delta(\omega) + (\frac{dG(\omega)}{d\theta})^2 \frac{1}{1 + \frac{S(\omega)}{N(\omega)} G(\omega)}] \, d\omega \tag{45}$$

where

$$\delta(\omega) \equiv 2\{G(\omega)\frac{d\underline{V}^*(\omega)}{d\theta} Q^{-1}(\omega) \frac{d\underline{V}(\omega)}{d\theta} - \left|\underline{V}^*(\omega)Q^{-1}(\omega) \frac{d\underline{V}(\omega)}{d\theta}\right|^2\} \tag{46}$$

This rather cumbersome expression will turn out to be of only minor importance in the end.

We now turn to the numerator of Eq. (44). Decomposing K into single frequency blocks K_j and proceeding exactly as in Eqs. (25)-(33) we obtain immediately (once again suppressing the argument ω_j)

$$K_j^{-1} \frac{\partial K_j}{\partial \omega_o} = \frac{\frac{1}{N} \frac{dS}{d\omega_o}}{1 + \frac{S}{N}G} Q^{-1}P \tag{47}$$

To compute $K_j \frac{\partial K_j}{\partial \theta}$ we recall that the source bearing θ affects the spatial covariance matrix P of the signal, but not its spectrum S. Hence

$$K_j^{-1}\frac{\partial K_j}{\partial \theta} = (S\ P + N\ Q)^{-1}\ S\ \frac{dP}{d\theta} \tag{48}$$

Using the matrix identity (31) one obtains after a few steps of algebra

$$K_j^{-1}\frac{\partial K_j}{\partial\theta} = \frac{S}{N}\left[I - \frac{\frac{S}{N}}{1+\frac{S}{N}G}Q^{-1}P\right]Q^{-1}\frac{dP}{d\theta} \tag{49}$$

Now combining Eqs. (47) and (49):

$$K_j^{-1}\frac{\partial K_j}{\partial\omega_o}\ K_j^{-1}\frac{\partial K_j}{\partial\theta} =$$

$$\frac{S}{N^2}\ \frac{\frac{dS}{d\omega_o}}{1+\frac{S}{N}G}\left[Q^{-1}P - \frac{\frac{S}{N}}{1+\frac{S}{N}G}(Q^{-1}P)^2\right]Q^{-1}\frac{dP}{d\theta} \tag{50}$$

However

$$(Q^{-1}P)^2 = Q^{-1}\underline{V}\ \overbrace{\underline{V}^*Q^{-1}\underline{V}}^{G}\ \underline{V}^* = GQ^{-1}P \tag{51}$$

Hence

$$K_j^{-1}\frac{\partial K_j}{\partial\omega_o}\ K_j^{-1}\frac{\partial K_j}{\partial\theta} = \frac{S}{N^2}\ \frac{\frac{dS}{d\omega_o}}{(1+\frac{S}{N}G)^2}\ Q^{-1}PQ^{-1}\frac{dP}{d\theta} \tag{52}$$

Eq. (44) requires us to compute the trace of this quantity and sum over the frequency index j:

$$\mathrm{Tr}\left\{K_j^{-1}\frac{\partial K_j}{\partial\omega_o}K_j^{-1}\frac{\partial K_j}{\partial\theta}\right\} = \frac{S}{N^2}\ \frac{\frac{dS}{d\omega_o}}{(1+\frac{S}{N}G)^2}\ \mathrm{Tr}(Q^{-1}PQ^{-1}\frac{dP}{d\theta}) \tag{53}$$

Now

$$\mathrm{Tr}(Q^{-1}PQ^{-1}\frac{dP}{d\theta}) = \mathrm{Tr}[Q^{-1}\underline{V}\underline{V}^*Q^{-1}(\frac{d\underline{V}}{d\theta}\underline{V}^* + \underline{V}\frac{d\underline{V}^*}{d\theta})]$$

$$= \mathrm{Tr}(\underbrace{\underline{V}^*Q^{-1}\underline{V}}_{G}\ \underline{V}^*Q^{-1}\frac{d\underline{V}}{d\theta}) + \mathrm{Tr}[Q^{-1}\underline{V}\ \underbrace{\underline{V}^*Q^{-1}\underline{V}}_{G}\frac{d\underline{V}^*}{d\theta}) \tag{54}$$ [1]

It follows that

$$\mathrm{Tr}(Q^{-1}PQ^{-1}\frac{dP}{d\theta}) = G\ \mathrm{Tr}(\underline{V}^*Q^{-1}\frac{d\underline{V}}{d\theta} + \frac{d\underline{V}^*}{d\theta}Q^{-1}\underline{V}) = G\ \mathrm{Tr}[\frac{d}{d\theta}\underline{V}^*Q^{-1}\underline{V}] = G\frac{dG}{d\theta} \tag{55}$$ [2]

---

[1] Using the invariance of the trace under rotation of terms in its argument.

[2] Using the fact that Q is independent of θ.

Finally, substituting Eq. (55) into Eq. (53) and evaluating the j sum as an integral, we obtain

$$\mathrm{Tr}\left(K^{-1}\frac{\partial K}{\partial\omega_o}K^{-1}\frac{\partial K}{\partial\theta}\right) = \int_o^\infty \frac{\frac{S^2(\omega)}{N^2(\omega)}G(\omega)}{\left[1+\frac{S(\omega)}{N(\omega)}G(\omega)\right]^2}\,\frac{\frac{dS(\omega)}{d\omega_o}}{S(\omega)}\,\frac{dG(\omega)}{d\theta}\,d\omega \tag{56}$$

Combining Eqs. (35), (45) and (56) one can now write down a formal expression for the coupling factor ρ of Eq. (44). However, the information of primary interest is furnished by Eq. (56) alone. Specifically, the integrand of (56) contains the factor $dG(\omega)/d\theta$. If, for example, the noise is spatially incoherent, so that $G(\omega) = M$, $dG/d\omega = 0$ and hence $\rho = 0$. In this important case there is no coupling and hence, according to Eq. (43), no degradation of the frequency estimate due to lack of prior information concerning the bearing.[1] More generally, the coupling between frequency and bearing estimation is weak as long as the array gain is only weakly dependent on bearing. In practice this is apt to be the case for noise fields which are more or less isotropic or, to a lesser extent, for any noise field which does not contain strongly directional components.[2]

3. Narrowband Gaussian signal. TW << 1.

Thus far we have only considered observation times large compared with the inverse bandwidth of our narrowband Gaussian siganl. From a practical point of view this is undoubtedly the most interesting case. However, it is certainly true that during the initial moments of any observation TW

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[1]Since the expression for $D^2(\hat{\theta})$ is analogous to Eq. (43) the converse is also true: Lack of prior information concerning ω_o does not degrade the bearing estimate.

[2]Note that array geometry also enters into $G(\omega)$. For a linear array and an isotropic noise field, for instance, $dG/d\omega = 0$ for a broadside target but becomes large within one or two beamwidths of endfire. A spherically symmetrical array in a similar noise field would yield $dG/d\omega = 0$ for all bearings.

is small. In certain situations (e.g. large signal to noise ratio) we might perhaps be fortunate enough to obtain the desired frequency information without ever reaching the large TW mode of operation. A brief treatment of the problem therefore appears in order.

Before beginning this discussion we must be aware of some fundamental limitations. A sample function of a narrowband Gaussian process looks essentially like a sinusoid of slowly varying frequency and amplitude. For times short compared with the inverse bandwidth its frequency and amplitude do not change significantly. All our estimator can do is to establish the present frequency. It has no information about the range of frequencies which will ultimately be traversed and can therefore not establish a center frequency. To make a center frequency measurement one requires a statistically significant sample of the random process, hence an observation time satisfying TW >> 1. The short term measurement is meaningful only if we are not concerned with the frequency fluctuations, i.e. if we do not seek an accuracy greater than the bandwidth of the narrowband process. This might be an acceptable condition if the bandwidth of the source is extremely narrow.

Assuming that we do wish to make the short time frequency estimate, we are faced with the problem of estimating the parameter ω_1 of a signal s(t) given by

$$s(t) = A \sin(\omega_1 t - \phi) \tag{57}$$

This is precisely the problem posed in section (1) with one exception: The amplitude A is an additional unknown parameter. One must therefore deal with the 3 x 3 Fisher information matrix of the triple(ω_1, ϕ, A). Computations proceed exactly as in section (1) and one finds (for the same signal level A and noise density N_1 at each sensor and for $T\omega_1 >> 1$).

$$
J = \begin{array}{ccc}
\omega & \phi & A \\
\left[\begin{array}{c|c|c}
\frac{M}{3}\frac{A^2}{N_1}\left[(t_1+T)^3 - t_1^3\right] & -\frac{M}{2}\frac{A^2}{N_1}\left[(t_1+T)^2 - t_1^2\right] & 0 \\
\hline
-\frac{M}{2}\frac{A^2}{N_1}\left[(t_1+T)^2 - t_1^2\right] & M\frac{A^2}{N_1}T & 0 \\
\hline
0 & 0 & \frac{M}{N_1}T
\end{array}\right]
\end{array} \tag{58}
$$

The amplitude estimate is evidently uncoupled from the other two and one obtains

$$
D^2(\hat{\omega}_1) = \frac{12}{T^3 M \frac{A^2}{N_1}} \tag{59}
$$

This is identical with Eq. (17). It is interesting to observe that the short time frequency measurement does, indeed, improve with T^3 until the inherent randomness of the process asserts itself (when $TW \to 1$). It is also important to keep in mind that Eq. (59) depends on A^{-2}. In the short term observation of the narrowband process, A is a random variable. We may be lucky and encounter a large value of A in the observation interval, enabling us to make an accurate short time frequency estimate. On the other hand, we may be unlucky and encounter a period of very low signal level so that little useful information can be extracted and we are forced into the use of larger observation intervals.[1]

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[1] The parameter A has a Rayleigh distribution, hence values near zero are not exceedingly unlikely.

III. Differential Doppler Estimation

When data are gathered at several widely separated subarrays, there may be differential Doppler shifts between the various subarray outputs. These provide information concerning source heading and velocity and are therefore of considerable practical importance. As in the case of single frequency estimation we distinguish between sinusoidal signals and narrowband Gaussian signals with $TW \gg 1$. Since narrowband Gaussian signals with $TW \ll 1$ are effectively sinusoids, we ignore that category.

In much of the development concerning frequency measurement without differential Doppler shift we allowed the array to consist of an arbitrary number of elements in arbitrary geometrical configuration. In each instance we found that the array contributed to frequency measurement only through enhancement of the effective signal to noise ratio. The output of the optimal beamformer contained all available information concerning the frequency or center frequency. Beamforming prior to frequency measurement is therefore not only a practical convenience, but it is actually the best possible procedure. We have pointed out before that the beamformer output may be treated as the output of a single "equivalent" sensor with improved signal to noise ratio. From here on we shall consider each subarray as such an equivalent sensor and shall use the terms "subarray" and "sensor" interchangeably.

1. Sinusoidal signal

Differential Doppler estimation from sinusoidal signals is basically the same problem as single frequency estimation. Suppose that we have two

[1] The "optimal" beamformer is that which maximizes the output signal to noise ratio.

subarrays with beamformer outputs $x_1(t)$ and $x_2(t)$ given by the equations

$$x_1(t) = A \sin [\omega_o(t-\tau_1) - \phi_1] + n_1(t) \tag{60}$$

$$x_2(t) = A \sin[(\omega_o+\Delta\omega)(t-\tau_2) - \phi_2] + n_2(t) \tag{61}$$

The unknown parameters are now the frequency ω_o, the differential Doppler shift $\Delta\omega$, and the two phase angles ϕ_1 and ϕ_2.

A few words of explanation may be in order concerning the need to introduce a second unknown phase angle. Suppose we had chosen $\phi_1 = \phi_2 = \phi$ as in part II. Except for small differential delays $\tau_1 - \tau_2$ [which are discarded in steps such as the transition from (12) to (13)] we now know that the signal components of $x_1(t)$ and $x_2(t)$ have zero crossings which are aligned at $t = 0$. By observing their relative displacement at t_1, the actual starting point of the observation interval, we could draw strong inferences concerning $\Delta\omega$. Indeed, formal computation of the estimation error yields

$$D^2(\Delta\hat{\omega})\Big|_{\phi_1=\phi_2=\phi} = \frac{6}{(t_1+T)^3 - t_1^{\,3}} \tag{62}$$

The formally correct but practically spurious t_1 dependence is precisely the same as that which motivated the introduction of the unknown phase angle ϕ in section II-1.

Once this pitfall is properly avoided by the use of separate ϕ_1 and ϕ_2 the computation of the Fisher information matrix for $(\omega_o, \Delta\omega, \phi_1, \phi_2)$ can procede exactly as in Section II-1. The result is

$$J = \left[\begin{array}{c|c|c|c} \frac{2}{3}C & \frac{1}{3}C & -\frac{1}{2}D & -\frac{1}{2}D \\ \hline \frac{1}{3}C & \frac{1}{3}C & 0 & -\frac{1}{2}D \\ \hline -\frac{1}{2}D & 0 & T & 0 \\ \hline -\frac{1}{2}D & -\frac{1}{2}D & 0 & T \end{array}\right] \tag{63}$$

where

$$C = \frac{A^2}{N_1}[(t_1+T)^3 - t_1{}^3], \; D = \frac{A^2}{N_1}[(t_1+T)^2 - t_1{}^2] \tag{64}$$

After a tedious but straightforward matrix inversion one finds

$$D^2(\Delta\hat{\omega}) = \frac{24}{\frac{A^2}{N_1} T^3} \tag{65}$$

and

$$D^2(\hat{\omega}_o) = \frac{12}{\frac{A^2}{N_1} T^3} \tag{66}$$

Comparing Eq. (66) with Eq. (17) [M = 2] we note that the form is the same, but that 3db of performance have been lost through introduction of the second unknown phase angle. The error in the $\Delta\omega$ estimate is twice as large because there is only one differential frequency whereas two opportunities exist to measure ω_o. We also note in passing that Eq. (62) becomes identical with Eq. (65) if one chooses $t_1 = T/2$, the worst value according to Eq. (62).[1] We note further that computation of differential Doppler shift from separate, independent frequency measurements at the two subarrays leads to precisely the same mean square error as that given by Eq. (66). Coherent processing of the subarray output provides no advantage.

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[1]A similar observation could be made about all computations in section II-1 in which the time origin is an apparent factor.

2. Narrowband Gaussian signal. TW >> 1. Two Subarrays.

A. The Data Covariance Matrix and Its Inverse.

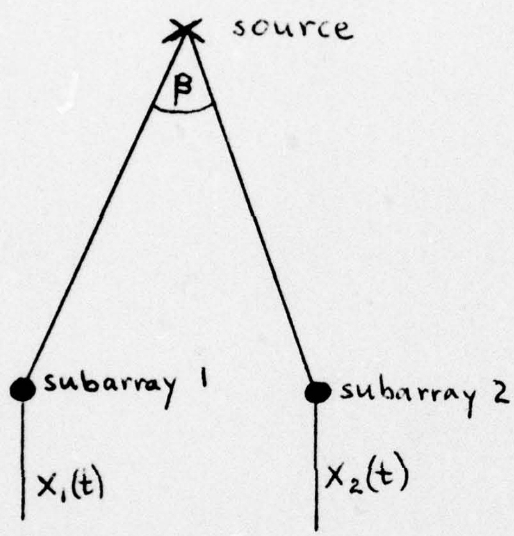


Fig. 2

We are now dealing with a source radiating a narrowband stationary Gaussian process with spectrum $S(\omega)$, symmetrical about a center frequency ω_o. Two receiving subarrays [Fig. 2] subtend a sufficient angle β at the source so that the Doppler shifts of the received signal components are not identical. We are concerned only with observation times satisfying TW >> 1 because only in that case are we dealing with a problem significantly different from that discussed in the previous section.

Our data vector, as in section II-2, is Gaussian. The basic expression for estimation error therefore remains Eq. (25). At this point, however, the strict parallel with the zero differential Doppler problem ends. In section II-2 we were able to block-diagonalize the data covariance matrix K by working with Fourier coefficients. We shall still find it useful to work with Fourier coefficients, but we no longer achieve the immediate block-diagonalization because differential Doppler shift can cause correlation between Fourier coefficients at different frequencies. Our first task is therefore computation of the general data covariance matrix K.

The data vector consists of the Fourier coefficients

$$X_i(\omega_n) = \int_{-\frac{T}{2}}^{\frac{T}{2}} x_i(t)e^{-j\omega_n t}\, dt \qquad i = 1,2 \tag{67}$$

The elements of the covariance matrix are

$$E\{X_i(\omega_n)X_j^*(\omega_\ell)\} = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_{-\frac{T}{2}}^{\frac{T}{2}} d\sigma\, R_{ij}(t,\sigma)e^{-j(\omega_n t - \omega_\ell \sigma)} \tag{68}$$

where

$$R_{ij}(t,\sigma) = E\{x_i(t)x_j(\sigma)\} \qquad i = 1,2 \tag{69}$$

is the temporal correlation of the received time functions. E{ } stands for the expectation of the bracketed quantity.

When i=j, Eq. (69) is the autocorrelation function of $x_i(t)$ and its computation from the signal and noise spectra is a simple matter. We therefore concentrate on the case i=1, j=2.

The signal components of $x_1(t)$ and $x_2(t)$ are stationary Gaussian processes and can therefore be represented by series of the form [5]

$$s_1(t) = \sum_{q=1}^{N\to\infty} c_q \cos[(\omega_q + \omega_D)t - \phi_q] \tag{70}$$

$$s_2(t) = \sum_{n=1}^{N\to\infty} c_q \cos[(\omega_q + \omega_D + \Delta\omega)(t - \tau_o) - \phi_q] \tag{71}$$

The ϕ_n are statistically independent random variables, each uniformly distributed over (0, 2π). c_n specifies the amplitude of the q^{th} sinusoid in terms of the known signal spectrum $S(\omega)$

$$\frac{c_q^2}{2} = S(\omega_q)\,\delta\omega, \qquad \delta\omega = \frac{2\pi}{T_o} \tag{72}$$

T_o is a very large but otherwise arbitrary interval for which (70) and (71) are defined. For our purposes we require only $T_o >> T$. The frequency components ω_q are given in terms of $\delta\omega$ by the equation

$$\omega_q = q\delta\omega = \frac{2\pi q}{T_o} \tag{73}$$

ω_D is the Doppler shift at subarray 1, $\Delta\omega$ the differential Doppler shift between the two subarrays and τ_o the differential delay of the signal.

The representations (70) and (71) imply a subtle but physically important assumption: Each component of $x_1(t)$ [and similarly $x_2(t)$] is Doppler shifted by the same amount. Actually the higher frequency components are shifted more, thus resulting in a slight distortion of the spectrum which is being ignored. The magnitude of this effect is bounded above by the Doppler shift of the maximum modulation frequency (signal bandwidth). In practice, our differential Doppler measurement will depend on the coherence of $s_1(t)$ and a down-shifted version of $x_2(t)$. That coherence will be affected seriously when the observation time T exceeds a period of the differential Doppler shift of the highest modulation frequency. Thus if σ is the signal bandwidth in rad/sec, Δv the difference of the source radial velocities measured at the two subarrays, and c the velocity of sound, our computation is valid for

$$T \frac{\Delta v}{c} \sigma << 1 \tag{74}$$

For values of T above this bound the notion of differential Doppler shift is no longer meaningful. One would have to estimate Δv directly, implying the use of true time compression rather than mere frequency shift.

Before proceeding with the computation we should clarify one additional point. In section III-1 we were forced to introduce the random phase angle ϕ_2 in order to eliminate the spurious dependence of Eq. (62) on the

time origin. It is tempting, but inappropriate, to introduce a similar random phase angle into Eq. (71). To do so would mean shifting each frequency component of (71) by the same angle. Such a procedure changes the waveshape and therefore affects the correlation between (70) and (71). The effect becomes serious as soon as TW > 1, precisely the region in which we wish to carry out our analysis. This leaves us with the problem of time origin dependence. We shall resolve it initially by choosing the time origin at the center of the observation interval ($t_1 = \frac{T}{2}$), the choice which yielded the correct result in the case of sinusoidal signals. Later we shall take the more fundamental approach of allowing τ_o to be unknown, thus eliminating the special significance of the time origin. We shall find that introduction of this additional unknown parameter does not change the estimation error of $\Delta\omega$, thus confirming the validity of the earlier result.

In much of the subsequent development computational details become quite cumbersome. They are therefore relegated to a series of appendices of which the first, Appendix A, deals with the derivation of $R_{ij}(t,\sigma)$ from Eqs. (69), (70) and (71). According to Eq. (A-3)

$$R_{12}(t,\sigma) = \int_0^\infty d\omega S(\omega)\cos[(\omega+\omega_D)(t-\sigma) - \Delta\omega\sigma + (\omega+\omega_D+\Delta\omega)\tau_o] \tag{75}$$

We note that our spectrum $S(\omega)$ is so defined that

$$\int_0^\infty S(\omega)d\omega = \text{Average signal power} \tag{76}$$

Only the signal contributes to R_{12} because the noises at the two subarrays are assumed to be uncorrelated.

Perhaps the most striking feature of Eq. (75) is the fact that it depends on both σ and t, not only on their difference. The source signal and the signals received at each subarray are stationary random processes, but they

are not stationarily correlated because of the differential Doppler shift. This is at the core of the difference between center frequency estimation and differential Doppler estimation. We shall find that the periodic nature of the non-stationarity can be exploited to obtain an essentially coherent estimate of $\Delta\omega$.

The next step is the computation of the data covariance matrix K from Eq. (68). Details are contained in Appendix B. Assuming only TW >> 1 the general result is [from Eqs. (B-19) (B-20) and (B-21)]:

$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} =$$

$$\pi T \frac{\sin(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}{(\omega_\ell-\omega_n-\Delta\omega)\ \frac{T}{2}} S\left(\frac{\omega_\ell+\omega_n-\Delta\omega}{2}-\omega_D\right) e^{j\frac{\tau_o}{2}(\omega_n+\omega_\ell+\Delta\omega)} \tag{77}$$

$$E\{X_1(\omega_n)X_1^*(\omega_\ell)\} = \begin{cases} \pi T[S(\omega_n-\omega_D) + N_1(\omega_n)] & n = \ell \\ 0 & n \neq \ell \end{cases} \tag{78}$$

$$E\{X_2(\omega_n)X_2^*(\omega_\ell)\} = \begin{cases} \pi T[S(\omega_n-\omega_D-\Delta\omega) + N_2(\omega_n) & n = \ell \\ 0 & n \neq \ell \end{cases} \tag{79}$$

The matrix K specified by Eqs. (77)-(79) is cumbersome because the cross-correlation for Fourier coefficients associated with different sub-arrays [Eq. (77)] is, in general, different from zero for all combinations of n and ℓ. Considerable simplification results if values of $\Delta\omega$ are confined to the discrete set

$$\Delta\omega = \frac{2\pi k}{T}, \quad k \text{ an integer} \tag{80}$$

Since, by definition

$$\omega_n = \frac{2\pi n}{T},\ \omega_\ell = \frac{2\pi\ell}{T} \qquad n, \ell \text{ integers} \tag{81}$$

one obtains

$$\frac{\sin(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}{(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}} = \frac{\sin(\ell-n-k)\pi}{(\ell-n-k)\pi} = \begin{cases} 1 & \ell = n+k \\ 0 & \ell \neq n+k \end{cases} \tag{82}$$

Hence Eq. (77) becomes

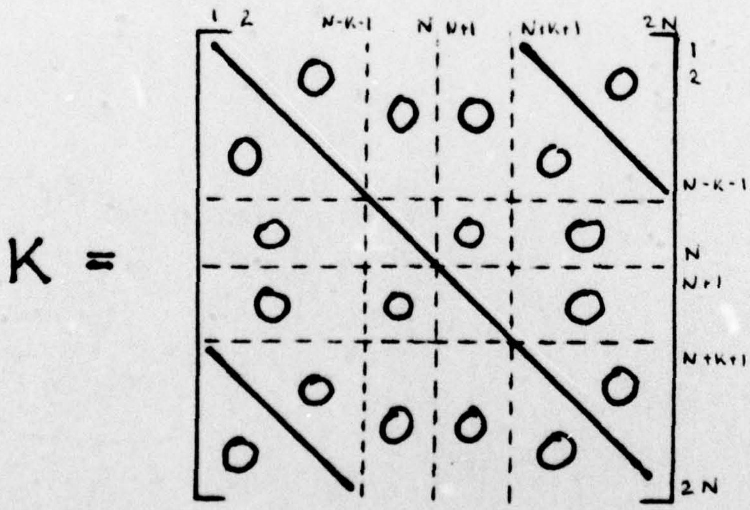
$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = \begin{cases} \pi T\ S(\omega_n-\omega_D)\ e^{j\tau_o(\omega_n+\Delta\omega)} & \ell = n+k \\ 0 & \ell \neq n+k \end{cases} \tag{83}$$

The assumption that $\Delta\omega$ takes on only a discrete set of values $2\pi k/T$ should be of small practical importance. Since TW >> 1, the allowed differential Doppler shifts are separated by much less than the signal bandwidth. It seems unlikely that estimator performance for this rather dense set of $\Delta\omega$ values should not be generally representative.

For later manipulations we shall find it convenient to arrange the data vector as follows

$$\underline{x} = [X_1(\omega_1)X_1(\omega_2)\ldots X_1(\omega_N)X_2(\omega_1)X(\omega_2)\ldots X_2(\omega_N)]^T \tag{84}$$ [1]

Then the form of the covariance matrix K is



(85)

---

[1]The complex Fourier representation requires both positive and negative frequencies. However, $X_1(-\omega_n) = X_1^*(\omega_n)$. Thus the negative frequency coefficients do not contribute new information and can therefore be omitted.

The solid diagonal lines indicate the only non-zero entries in the matrix. Coupling between the two subarray outputs exists only at frequencies separated by the differential Doppler shift.

Because of the sparse form of Eq. (85) the calculation of K^{-1} is not difficult. Computational details are given in Appendix C and the result is [for spectrally flat noise of power level N_o at each subarray]

$$K^{-1} = \frac{1}{\pi T} \begin{array}{c} \begin{array}{cccc} \qquad\qquad\qquad & K & K & \qquad\qquad\qquad \end{array} \\ \left[\begin{array}{c:c:c:c} \begin{matrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_{N-K} \end{matrix} & 0 & 0 & \begin{matrix} -D_1 e^{j\rho_1} & & & \\ & -D_2 e^{j\rho_2} & & \\ & & \ddots & \\ & & & -D_{N-K} e^{j\rho_{N-K}} \end{matrix} \\ \hdashline 0 & \begin{matrix} \frac{1}{N_o} & & \\ & \ddots & \\ & & \frac{1}{N_o} \end{matrix} & 0 & 0 \\ \hdashline 0 & 0 & \begin{matrix} \frac{1}{N_o} & & \\ & \ddots & \\ & & \frac{1}{N_o} \end{matrix} & 0 \\ \hdashline \begin{matrix} -D_1 e^{-j\rho_1} & & & \\ & -D_2 e^{-j\rho_2} & & \\ & & \ddots & \\ & & & -D_{N-K} e^{-j\rho_{N-K}} \end{matrix} & 0 & 0 & \begin{matrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_{N-K} \end{matrix} \end{array}\right] \end{array} \begin{array}{l} \\ \\ \} K \\ \} K \\ \\ \end{array} \qquad (86)$$

where

$$B_n = \frac{S(\omega_n-\omega_D) + N_o}{2\ N_o\ S(\omega_n-\omega_D) + N_o^2} \tag{87}$$

$$D_n = \frac{S(\omega_n - \omega_D)}{2\ N_o\ S(\omega_n-\omega_D) + N_o^2} \tag{88}$$

and

$$\rho_n = (\omega_n + \Delta\omega)\ \tau_o \tag{89}$$

B. Estimation of Differential Doppler Shift

The next step in the evaluation of Eq. (25) requires calculation of the derivative $(dK/d\alpha)$. For the moment we shall assume that $\Delta\omega$ is the only unknown parameter. We must therefore calculate the elements of $[dK/d(\Delta\omega)]$.

While we are ultimately concerned only with the discrete set of differential Doppler shifts $\Delta\omega = 2\pi k/T$, we cannot start with Eq. (83) because it does not describe the $\Delta\omega$ dependence near the point of interest, which is required for the computation of the derivative. We must therefore return to Eqs. (77)-(79), differentiate with respect to $\Delta\omega$ and then evaluate at $\Delta\omega = 2\pi k/T$. The operation is trivial for Eqs. (78) and (79). The computation for Eq. (77) is shown in Appendix D.

$$\frac{d}{d(\Delta\omega)} E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = \begin{cases} \frac{\pi T^2}{2}[j\frac{\tau_o}{T}\ S(\omega_n-\omega_D) - \frac{1}{T}S'(\omega_n-\omega_D)]e^{j2\pi\frac{\tau_o}{T}} & \ell = n+k \\ -\frac{T^2}{2}\frac{(-1)^{\ell-n-k}}{\ell-n-k}\ S[\frac{\pi}{T}(\ell+n-k)-\omega_D]e^{j\frac{\tau_o}{T}(n+\ell+k)} & \ell \neq n+k \end{cases} \tag{90}$$

$$\frac{d}{d(\Delta\omega)} E\{X_1(\omega_n)X_1^*(\omega_\ell)\} = 0 \tag{91}$$

$$\frac{d}{d(\Delta\omega)}\, E\{X_2(\omega_n)X_2^*(\omega_\ell)\} = \begin{cases} -\pi T\; S'(\omega_n-\omega_D-\Delta\omega) & n = \ell \\ 0 & n \neq \ell \end{cases} \tag{92}$$

The various components of K have very different orders of magnitude. Since $\tau_o << T$ the τ_o dependent term in the first line of Eq. (90) is several orders or magnitude smaller than the term in the second line. The physical interpretation of $(2\pi/T)S'(\omega_n-\omega_D)$ is the change in S over an interval of $(2\pi/T)$ rad/sec near $\omega_n-\omega_D$. For $TW >> 1$ and smoothly varying spectra this is a very small fraction of the value of S in the same neighborhood. Both terms in the first line of Eq. (90) are therefore orders of magnitude smaller than the term in the second line and can be ignored without incurring significant error. The only non-zero term in Eq. (92) is of the same order as the S' term in Eq. (90). It will therefore be ignored also.[1] This leaves the following simplified version of $(dK/d(\Delta\omega))$.

$$\frac{d}{d(\Delta\omega)}\, E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = \begin{cases} 0 & \ell = n+k \\ -\frac{T^2}{2}\,\frac{(-1)^{\ell-n-k}}{\ell-n-k}\, S[\frac{\pi}{T}(\ell+n-k)-\omega_D]\,e^{j\frac{\tau_0}{T}\pi(n+\ell+k)} & \ell \neq n+k \end{cases} \tag{93}$$

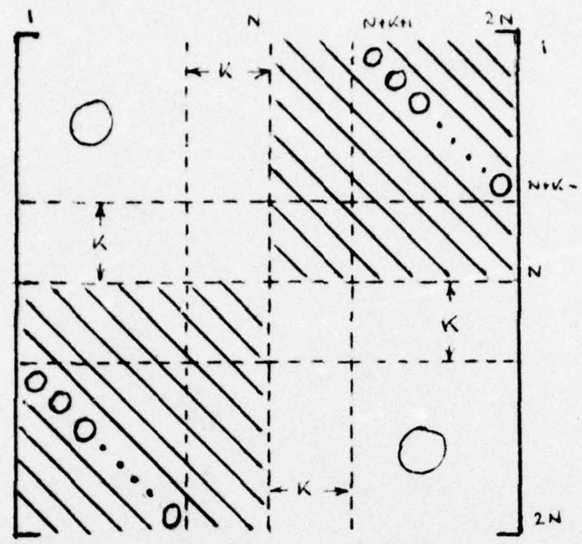
$$\frac{d}{d(\Delta\omega)}\, E\{X_1(\omega_n)X_1^*(\omega_\ell)\} = \frac{d}{d(\Delta\omega)}\, E\{X_2(\omega_n)X_2^*(\omega_\ell)\} = 0 \tag{94}$$

The form of the complete $dK/d(\Delta\omega)$ matrix is shown in Eq. (95).

---

[1]Note that Eq. (92) characterizes our ability to estimate $\Delta\omega$ from observations at the second subarray alone. This is what we are discarding when we make the proposed approximation.

$$\frac{dK}{d(\Delta\omega)} =$$



(95)

As in Eq. (85), the solid diagonal lines indicate the only non-zero entries in the matrix. We note that the factor $(\ell-n+k)^{-1}$ in Eq. (93) causes the magnitude of these non-zero entries to vary inversely with their distance from the diagonal line of zeros.

We must now combine Eqs. (86) and (95) to obtain $\mathrm{Tr}\{(K^{-1}\frac{dK}{d\Delta(\omega)})^2\}$. Computational details are shown in Appendix E. From Eq. (E-21)

$$\mathrm{Tr}\ \{(K^{-1}\frac{dK}{d(\Delta\omega)})^2\} =$$

$$\frac{T^3}{4\pi^3}\sum_{r\neq 0}\frac{1}{r^2}\int_0^\infty \frac{N_0[S(\omega-\frac{\pi r}{T}-\omega_D)+S(\omega+\frac{\pi r}{T}-\omega_D)]+N_0^2}{[2N_0S(\omega-\frac{\pi r}{T}-\omega_D)+N_0^2][2N_0S(\omega+\frac{\pi r}{T}-\omega_D)+N_0^2]}\,S^2(\omega-\omega_D)\,d\omega \qquad (96)$$

Eq. (96) can be used for a cumbersome but rather precise computation of the $\Delta\omega$ estimation error. A much simpler form can be obtained by making an additional approximation: Because of the factor $1/r^2$ only small values of r contribute significantly to the sum. For small r, $S(\omega)$ will

not vary greatly over an interval of $\pi r/T$ and one can approximate

$S(\omega \pm \frac{\pi r}{T} - \omega_D) \approx S(\omega - \omega_D)$.[1] In that case

$$\mathrm{Tr}\{(K^{-1}\frac{dK}{d(\Delta\omega)})^2\} \simeq \frac{T^3}{2\pi^3}\left(\sum_{r=1}^{\infty}\frac{1}{r^2}\right)\int_0^{\infty}\frac{S^2(\omega-\omega_D)}{2N_0 S(\omega-\omega_D) + N_0^2}\,d\omega \tag{97}$$

The mean square estimation error is simply the inverse of this quantity. The r sum of Eq. (97) has the numerical value $\pi^2/6$ [6]. Hence

$$D^2(\Delta\hat{\omega}) = \frac{1}{\frac{T^3}{12\pi}\int_0^{\infty}\frac{S^2(\omega-\omega_D)}{2N_0 S(\omega-\omega_D) + N_0^2}\,d\omega} \tag{98}$$

C. Comparison of differential Doppler estimation with center frequency estimation and frequency estimation for sinusoidal signal.

Perhaps the most striking difference between differential Doppler estimation [Eq. (98)] and center frequency estimation [Eq.(35)] is the T dependence of the mean square error. $D^2(\Delta\hat{\omega})$ varies as T^{-3} whereas $D^2(\hat{\omega}_0)$ varies as T^{-1}. In this respect the differential Doppler estimate behaves like the frequency estimate of a sinusoidal signal [Eq.(19)]. This is not

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[1] The approximation is better than one might think at first glance. If the spectrum varies linearly over the significant range of $\pm\pi r/T$, the numerator of Eq. (96) is precisely $[2N_0S(\omega-\omega_D) + N_0^2]$. The denominator differs from $[2N_0S(\omega-\omega_D) + N_0^2]^2$ only by $4N_0^2[S(\omega - \frac{\pi r}{T} - \omega_D)S(\omega + \frac{\pi r}{T} - \omega_D) - S^2(\omega-\omega_D)]$. For large S/N the fractional error is therefore proportional to the square of the change in the spectral level divided by $S^2(\omega-\omega_D)$. For small S/N the approximation is even better.

unreasonable: Even though the signal is a Gaussian random process, the waveshapes received at the two subarrays are deterministically related and the differential Doppler estimate can therefore proceed on a coherent basis.[1] No such possibility exists for the center frequency estimate which is essentially a measurement of power distribution over frequency.

To explore further the apparent similarity between differential Doppler estimation and frequency estimation for a sinusoid we consider separately the cases of high and low signal to noise ratio in the signal band. From a practical point of view the high S/N case is probably the most interesting one. In any event, it is most directly comparable with the sinusoidal frequency estimation problem since the in-band signal to noise ratio is almost certainly high in the latter.[2]

a. High signal to noise ratio in the signal band.

When $S(\omega) >> N_0$ in the signal band, Eq. (98) can be approximated to a good degree of accuracy by the equation

$$D^2(\Delta\hat{\omega}) \simeq \frac{1}{\frac{T^3}{24\pi N_0}\int_0^\infty S(\omega)d\omega} = \frac{24\pi}{T^3 \frac{P_s}{N_0}} \tag{99}$$

P_s is the total signal power. We note that the error depends only on the total signal power, not on detailed spectral properties of the signal.

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[1] A word of caution is in order here. As pointed out earlier, we have assumed that each frequency component received at a subarray is Doppler shifted by the same amount. In reality the higher frequency components are shifted by slightly larger amounts. This causes waveshape distortion and if this distortion is not the same at both subarrays, there will be decorrelation. Time compression of one subarray output should compensate for this effect, but the analysis presented here does not deal with that problem.

[2] In the sinusoidal case the effective bandwidth is $2\pi/T$.

Eq. (99) is not yet directly comparable with Eq. (17). The noise spectral density N_1 in Eq. (17) was defined as the power per Herz, whereas N_0 in Eq. (99) is the power per radian/sec. Thus

$$N_1 = 2\pi N_0 \tag{100}$$

The sinusoidal signal power in Eq. (17) is

$$\frac{A^2}{2} = P_s \tag{101}$$

With this nomenclature Eq. (17) becomes

$$D^2(\hat{\omega}_0) = \frac{12\pi}{T^3 M \frac{P_s}{N_0}} \tag{102}$$

Since the two sensors (subarrays) employed in differential Doppler estimation afford only one opportunity for differential frequency measurement, one should probably choose M = 1 in Eq. (102) in order to obtain a fair comparison. With that understanding

$$\frac{D^2(\Delta\hat{\omega})}{D^2(\hat{\omega}_0)} = 2 \tag{103}$$

Thus the differential Doppler measurement for a narrowband Gaussian signal differs in mean square error from the frequency measurement for a sinusoid by a fixed factor of 2, independent of signal and noise properties.

Another and perhaps more meaningful comparison can be made between differential Doppler measurements using sinusoids and narrowband Gaussian signals respectively. From Eqs. (65) and (99) [using Eqs. (100) and (101)]

$$\frac{D^2(\Delta\hat{\omega})\big|_{\text{narrowband}}}{D^2(\Delta\hat{\omega})\big|_{\text{sinusoid}}} = 1 \tag{104}$$

The narrowband differential Doppler estimator is therefore equal in performance to an estimator supplied with sinusoidal signals.

b. Low signal to noise ratio in the signal band.

When the signal to noise ratio does not exceed unity even in the signal band, Eq. (98) can be approximated by

$$D^2(\Delta\hat{\omega}) \simeq \frac{1}{\frac{T^3}{12\pi N_0^2} \int_0^\infty S^2(\omega-\omega_D)d\omega} \tag{105}$$

In contrast with the high signal to noise ratio situation the estimation error now depends to some extent on the shape of the signal spectrum. As an example we use the spectral shape of Eq. (37), here repeated for reference

$$S(\omega) = S_0\{\exp - \frac{(\omega-\omega_0)^2}{2\sigma^2}\} \tag{106}$$

A simple computation now yields

$$D^2(\Delta\hat{\omega})\Big|_{\text{low } S/N} = \frac{12\sqrt{\pi}}{T^3 \sigma \frac{S_0^2}{N_0^2}} \tag{107}$$

To convert the high S/N result to a comparable form we note that

$$P_s = \int_0^\infty S_0 \exp\{-\frac{(\omega-\omega_0)^2}{2\sigma^2}\}d\omega = \sqrt{2\pi}\ \sigma\ S_0 \tag{108}$$

Then Eq. (99) becomes

$$D^2(\Delta\hat{\omega})\Big|_{\text{high } S/N} = \frac{24\sqrt{\pi}}{\sqrt{2}\ T^3 \sigma \frac{S_0}{N_0}} \tag{109}$$

The only significant difference between Eqs. (107) and (109) is the signal to noise ratio dependence. The differential Doppler estimator operates coherently in either regime [T^{-3} dependence], but its performance figure varies as the square of the signal to noise ratio for low S_0/N_0, as its first power for large S_0/N_0.

Finally we wish to make a quantitative comparison between the differential Doppler estimator and an estimator which obtains the differential Doppler shift from separate center frequency estimates at the two subarrays. Since the center frequency estimate depends on the signal spectrum we continue to work with the example of Eq. (106).

a. High signal to noise ratio in the signal band.

From Eq. (42) we obtain for the mean square error of the center frequency estimate at each subarray (M = 1)

$$D^2(\hat{\omega}_0) = \frac{3\pi}{2\sqrt{2}\,T} \frac{\sigma}{\left(\log \frac{S_0}{N_0}\right)^{3/2}} \tag{110}$$ [1]

When estimates at the two subarrays are subtracted to obtain differential Doppler shift, this mean square error is doubled. The direct differential Doppler estimate yields the mean square error given by Eq. (109). Hence

$$\frac{D^2(\Delta\hat{\omega})\Big|_{\text{subtraction of } \omega_0 \text{ estimates}}}{D^2(\Delta\hat{\omega})\Big|_{\text{direct estimate}}} = 0.222 \frac{S_0/N_0}{(\log S_0/N_0)^{3/2}} (T\sigma)^2 \qquad S_0/N_0 \gg 1 \tag{111}$$ [1]

[1] log ( ) means the logarithm to base e.

The most important feature of Eq. (111) is the factor $(T\sigma)^2$. σ is the signal bandwidth (in rad/sec). We therefore conclude that the direct estimate of $\Delta\omega$ is better than an estimate derived from separate center frequency measurements by a factor proportional to the square of the TW product. A second, though lesser, advantage of the direct measurement is its more rapid improvement with signal to noise ratio. Since the logarithmic dependence in Eq. (109) comes from the particular spectral form of Eq. (106), the signal to noise dependence could vary substantially and might well be less favorable to the direct measurement than suggested by Eq. (111).

b. Low signal to noise ratio in the signal band.

When $S_0/N_0 \ll 1$ the mean square error of the center frequency estimate is given by Eq. (39). For M = 1

$$D^2(\hat{\omega}_0) = \frac{\sqrt{\pi}\,\sigma}{2T\left(\frac{S_0}{N_0}\right)^2} \tag{112}$$

The mean square error of the direct differential Doppler estimate is given by Eq. (107). Hence

$$\frac{D^2(\Delta\hat{\omega})\Big|_{\text{subtraction of } \omega_0 \text{ estimates}}}{D^2(\Delta\hat{\omega})\Big|_{\text{direct measurement}}} = \frac{1}{12}(T\sigma)^2\,, \quad S_0/N_0 \ll 1 \qquad (113)^{1}$$

Once again the advantage of the direct measurement procedure is proportional to the square of the TW product. For $S_0/N_0 \ll 1$ the improvement is independent of the signal to noise ratio.

[1] Note that TW > 1 implies $T\sigma > 2\pi$, so that the right side of Eq. (113) is substantially in excess of unity even for very moderate values of TW.

D. Simultaneous estimation of differential Doppler shift and other parameters.

Thus far we have only considered differential Doppler estimation as an isolated problem, all other parameters being assumed known. In practice, this is almost certainly unrealistic. Parameters such as source bearing and range, signal center frequency and bandwidth are probably not known a priori and this lack of knowledge might degrade the differential Doppler estimate. The amount of any degradation depends on the coupling between the differential Doppler estimate and estimates of the other potentially unknown parameters. We now study this coupling on a pairwise basis.

The basic relations are Eqs. (43) and (44), rewritten here for the estimation of $\Delta\omega$.

$$D^2(\Delta\hat{\omega}) = \frac{D^2(\Delta\hat{\omega})\Big|_{\alpha \text{ known}}}{1 - \rho} \tag{114}$$

where

$$\rho = \frac{\left[\mathrm{Tr}\left(K^{-1}\frac{\partial K}{\partial \alpha}K^{-1}\frac{\partial K}{\partial(\Delta\omega)}\right)\right]^2}{\mathrm{Tr}\left[\left(K^{-1}\frac{\partial K}{\partial(\Delta\omega)}\right)^2\right]\mathrm{Tr}\left[\left(K^{-1}\frac{\partial K}{\partial \alpha}\right)^2\right]} \tag{115}$$

α is the second unknown parameter whose coupling with $\Delta\omega$ is under investigation. For our purposes it will be bearing, range, center frequency or signal bandwidth. Fortunately it turns out that we need not make a separate argument for each of these quantities.

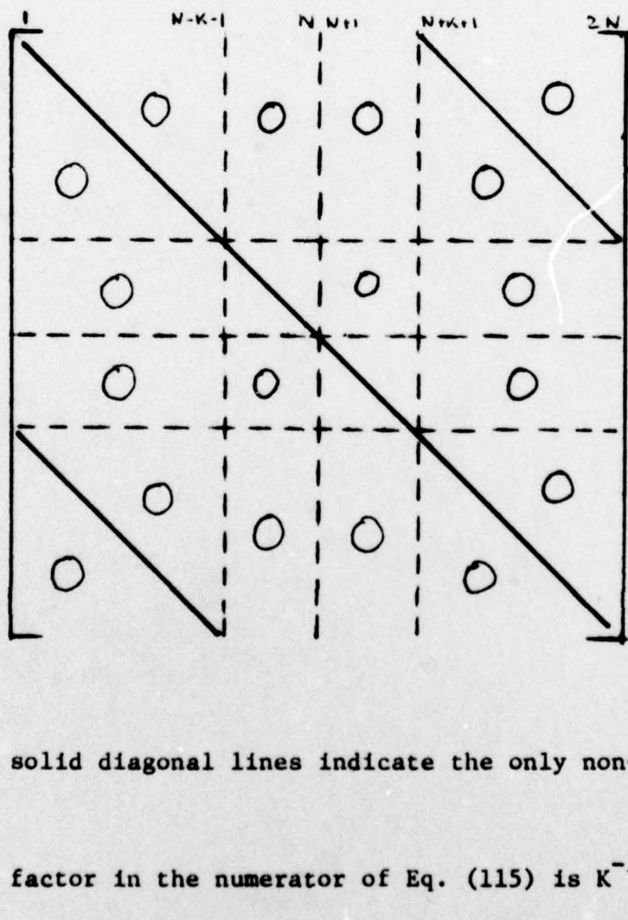
The key to a simpler argument is the manner in which the second parameter α enters the data covariance matrix K whose elements are given by Eqs. (77)-(79). Bearing and range affect only the delay τ_0. Signal center frequency and bandwidth appear only in the spectral function S( ). Differentiation with respect to these parameters is a simple procedure which does not

affect the multiplying factor

$$\frac{\sin(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}{(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}$$

in Eq. (77). When the derivative is evaluated at $\Delta\omega = 2\pi k/T$, k an integer, this factor causes the differentiated Eq. (77) to vanish unless $\ell = n+k$ Thus $\partial K/\partial\alpha$ retains the form of Eq. (85) for each of the parameters α under discussion. It is now a simple matter to demonstrate [see Appendix F for details] that $K^{-1}\frac{dK}{d\alpha}$ also has the same form.

$$K^{-1}\frac{\partial K}{\partial\alpha} = \qquad (116)$$

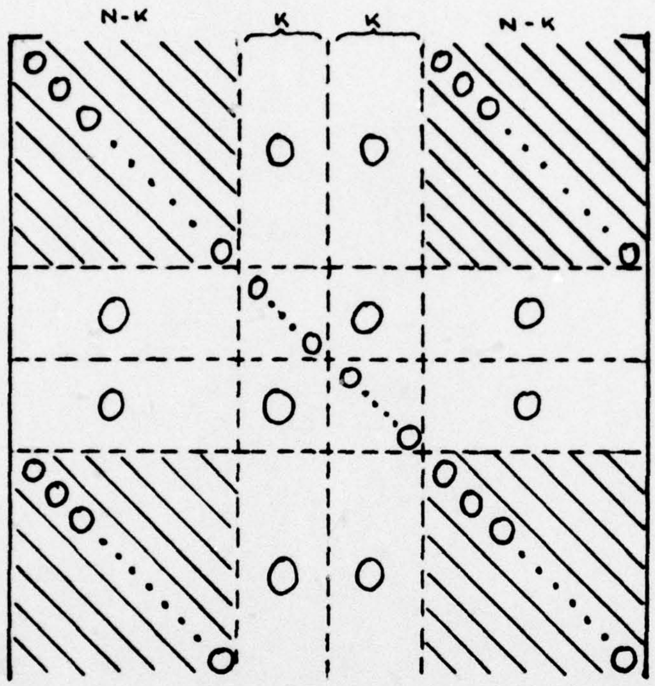


Once again the solid diagonal lines indicate the only non-zero entries in the matrix.

The other factor in the numerator of Eq. (115) is $K^{-1}\frac{\partial K}{\partial(\Delta\omega)}$ which was previously computed in Eq. (E-9). In that equation B is a diagonal matrix and A has zeros along its principal diagonal [see Eq. (E-8)]. Hence

$K^{-1} \frac{\partial K}{\partial(\Delta\omega)}$ has the form

$$K^{-1} \frac{\partial K}{\partial(\Delta\omega)} = \tag{117}$$



Comparing Eqs. (116) and (117) we note that (117) has zeros in all slots where (116) shows non-zero entries. From this and the symmetry of the two matrices it follows immediately that

$$Tr(K^{-1} \frac{\partial K}{\partial \alpha} K^{-1} \frac{\partial K}{\partial(\Delta\omega)}) = 0 \tag{118}$$

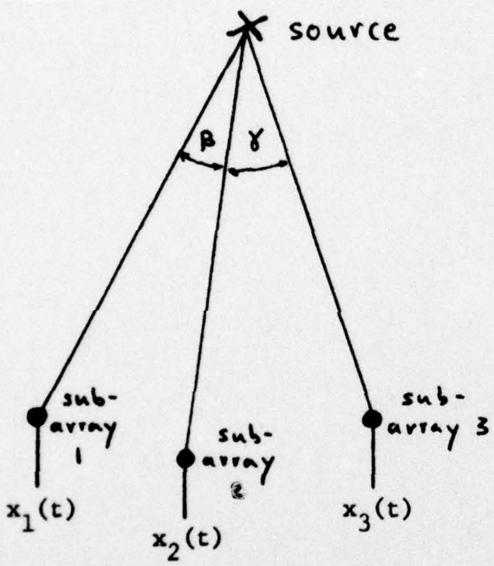
Thus $\rho = 0$ and from Eq. (114)

$$D^2(\Delta\hat{\omega}) = D^2(\Delta\hat{\omega})\Big|_{\alpha \text{ known}} \tag{119}$$

Differential Doppler estimation is therefore completely decoupled from the estimation of the other parameters under discussion. It is also apparent that this conclusion is not confined to the four specific secondary parameters

enumerated above. The same argument would apply to any parameter affecting only element to element delay or spectral shape of the signal. Differential Doppler shift is apparently very much in a class by itself and its lack of coupling to most other parameters makes it a particularly attractive feature to exploit in gathering information about source motion.

3. Narrowband Gaussian signal. TW >> 1. Three subarrays.



We now consider a receiving array composed of 3 subarrays which subtend sufficiently large angles β and γ at the source so that there can be measurable differential Doppler shifts. We deal only with the case of narrowband Gaussian signals and large TW products.

In complete analogy with Eqs. (70) and (71) the signal components of the waveshapes received at the three subarrays are now

$$s_1(t) = \sum_{q=1}^{N\to\infty} C_q \cos[\omega_q+\omega_D]t - \phi_q] \tag{120}$$

$$s_2(t) = \sum_{q=1}^{N\to\infty} C_q \cos[(\omega_q+\omega_D+\Delta_1)(t-\tau_1) -\phi_q] \tag{121}$$

$$s_3(t) = \sum_{q=1}^{N\to\infty} C_q \cos[(\omega_q+\omega_D+\Delta_2)(t-\tau_2) -\phi_q] \tag{122}$$

Δ_1 is the differential Doppler shift between subarrays 1 and 2 and τ_1 is the time delay for the same pair. Δ_2 is the differential Doppler shift between subarrays 1 and 3 and τ_2 is the corresponding time delay. All other symbols retain the definitions established in connection with Eqs. (70) and (71). Eqs. (120)-(122) also retain the earlier assumption that differential Doppler shifts on the modulation frequency are negligible.

We continue to represent the received data by Fourier coefficients. The data covariance matrix K now has dimension 3N x 3N and its elements are, in complete analogy with Eqs. (77)-(79):

$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = \pi T\frac{\sin(\omega_\ell-\omega_n-\Delta_1)\frac{T}{2}}{(\omega_\ell-\omega_n-\Delta_1)\frac{T}{2}} S(\frac{\omega_\ell+\omega_n-\Delta_1}{2} - \omega_D)e^{j\frac{\tau_1}{2}(\omega_n+\omega_\ell+\Delta_1)} \tag{123}$$

$$E\{X_1(\omega_n)X_3^*(\omega_\ell)\} = \pi T\frac{\sin(\omega_\ell-\omega_n-\Delta_2)\frac{T}{2}}{(\omega_\ell-\omega_n-\Delta_2)\frac{T}{2}} S(\frac{\omega_\ell+\omega_n-\Delta_2}{2} - \omega_D)e^{j\frac{\tau_2}{2}(\omega_n+\omega_\ell+\Delta_2)} \tag{124}$$

$$E\{X_2(\omega_n)X_3^*(\omega_\ell)\} = \pi T\frac{\sin(\omega_\ell-\omega_n-\Delta_2+\Delta_1)\frac{T}{2}}{(\omega_\ell-\omega_n-\Delta_2+\Delta_1)\frac{T}{2}} S(\frac{\omega_\ell+\omega_n-\Delta_2+\Delta_1}{2} -\omega_D-\Delta_1)e^{j\frac{\tau_2-\tau_1}{2}(\omega_n+\omega_\ell+\Delta_2-\Delta_1)} \tag{125}$$

$$E\{X_1(\omega_n)X_1^*(\omega_\ell)\} = \begin{cases} \pi T[S(\omega_n-\omega_D) + N(\omega_n)] & n = \ell \\ 0 & n \neq \ell \end{cases} \tag{126}$$

$$E\{X_2(\omega_n)X_2^*(\omega_\ell)\} = \begin{cases} \pi T[S(\omega_n-\omega_D-\Delta_1) + N(\omega_n)] & n = \ell \\ 0 & n \neq \ell \end{cases} \tag{127}$$

$$E\{X_3^*(\omega_n)X_3^*(\omega_\ell)\} = \begin{cases} T[S(\omega_n-\omega_D-\Delta_2) + N(\omega_n)] & n = \ell \\ 0 & n \neq \ell \end{cases} \tag{128}$$

As in the two subarray case, drastic simplification can be achieved by confining the actual values of differential Doppler shift to integral multiples of $2\pi/T$. Thus we will only allow the actual values

$$\Delta_1 = \frac{2\pi k}{T} \qquad k \text{ an integer} \tag{129}$$

and

$$\Delta_2 = \frac{2\pi h}{T} \qquad h \text{ an integer} \tag{130}$$

Then Eqs. (123)-(125) become

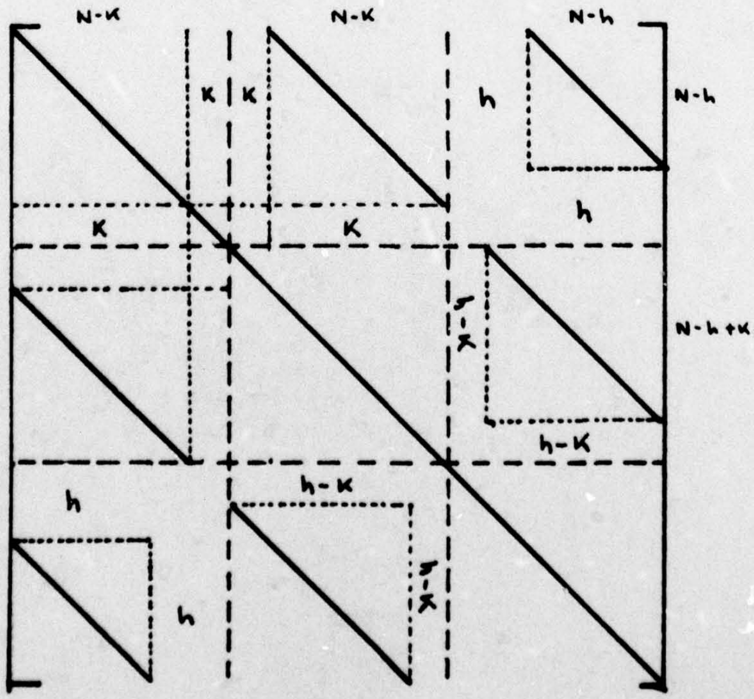
$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = \begin{cases} \pi T\ S(\omega_n-\omega_D)e^{j\tau_1(\omega_n+\Delta_1)} & \ell = n+k \\ 0 & \ell \neq n+k \end{cases} \tag{131}$$

$$E\{X_1(\omega_n)X_3^*(\omega_\ell)\} = \begin{cases} \pi T\ S(\omega_n-\omega_D)e^{j\tau_2(\omega_n+\Delta_2)} & \ell = n+h \\ 0 & \ell \neq n+h \end{cases} \tag{132}$$

$$E\{X_2(\omega_n)X_3^*(\omega_\ell)\} = \begin{cases} \pi T\ S(\omega_n-\omega_D-\Delta_1)e^{j(\tau_2-\tau_1)(\omega_n+\Delta_2-\Delta_1)} & \ell = n+h-k \\ 0 & \ell \neq n+h-k \end{cases} \tag{133}$$

Hence the data covariance matrix K has the form

K = (134)



In complete analogy with the argument leading to Eqs. (93) and (94) we obtain [again omitting terms of order T in comparison with those of order T^2]

$$\frac{\partial}{\partial\Delta_1} E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = \begin{cases} 0 & \ell = n+k \\ -\frac{T}{2}\frac{(-1)^{\ell-n-k}}{\ell-n-k} S[\frac{\pi}{T}(\ell+n-k)-\omega_D]e^{j\frac{\tau_1}{T}\pi(n+\ell-k)} & \ell \neq n+k \end{cases} \tag{135}$$

$$\frac{\partial}{\partial\Delta_1} E\{X_2(\omega_n)X_3^*(\omega_\ell)\} = \begin{cases} 0 & \ell = n+h-k \\ -\frac{T^2}{2}\frac{(-1)^{\ell-n-h+k}}{\ell-n-h+k} S[\frac{\pi}{T}(\ell+n-h+k)-\omega_D-\Delta_1]e^{j\frac{\tau_2}{T}\pi(n+\ell-h+k)} & \ell \neq n+h-k \end{cases} \tag{136}$$

Since X_1 and X_3 do not contain the parameter Δ_1

$$\frac{\partial}{\partial\Delta_1} E\{X_1(\omega_n)X_3^*(\omega_\ell)\} = 0 \tag{137}$$

Comparison of Eqs. (135)-(137) with Eqs. (131)-(133) reveals that the former are zero for all combinations of n and ℓ for which the latter are different from zero. The matrix $\partial K/\partial\Delta_1$ therefore has zero entries in all positions where Eq. (134) shows non-zero entries.[1]

Consider next the derivatives with respect to the delay parameters. $\partial K/\partial\tau_1$ and $\partial K/\partial\tau_2$ clearly have the same form as Eq. (134). An argument analogous to Appendix F shows that $K^{-1}\frac{\partial K}{\partial\tau_i}$, i = 1,2, also assumes this form.[2]

[1]The diagonal blocks of $\partial K/\partial\Delta_1$ are zero to the order of our approximation [see Eq. (94)].

[2]This is almost obvious by inspection. Since K can be converted into a block diagonal form of 3x3 matrices (Appendix G), K^{-1} and $\partial K/\partial\tau_i$ are similarly block diagonal and their product therefore has the same characteristic.

It follows immediately that

$$\mathrm{Tr}\left(K^{-1} \frac{\partial K}{\partial \tau_i} K^{-1} \frac{\partial K}{\partial \Delta_1}\right) = 0 \tag{138}$$

exactly as in the case of two subarrays. The lack of coupling between differential Doppler estimates and estimates of most other potentially unknown parameters (bearing, range, signal bandwidth, etc.) evidently carries over unchanged from the 2 subarray problem. In discussing differential Doppler estimation we can therefore assume, without loss of generality, that the relative delays τ_1 and τ_2 are known. If they are known, we can certainly introduce delay elements at the outputs of two of the subarrays which align the three signal components. Since these relative delays are reversible, their introduction cannot alter the performance of the optimal differential Doppler estimator. In evaluating differential Doppler estimation we can therefore, without loss of generality, set $\tau_1 = \tau_2 = 0$, thus greatly simplyfing Eqs. (131), (132), (135), and (136). Calculation of the mean square estimation error can now proceed in straightforward manner much as in the 2 subarray problems. Computational details are shown in Appendix H. With all other parameters (including Δ_2) known and with the same approximations as in the case of two subarrays

$$D^2(\hat{\Delta}_1)\Big|_{\Delta_2 \text{ known}} = \frac{1}{\dfrac{T^3}{6\pi}\displaystyle\int_0^\infty \frac{S^2(\omega-\omega_D)}{3\,N_0\,S(\omega-\omega_D) + N_0^2}\,d\omega} \tag{139}$$

Eq. (139) is somewhat misleading because the Δ_1 and Δ_2 estimates are not uncoupled and the assumption that Δ_2 is known whereas Δ_1 is not appears quite artificial. We must therefore study the degradation of the Δ_1 estimate due to lack of prior knowledge concerning Δ_2.

According to Eqs. (114) and (115) we require the coupling coefficient

$$\rho = \frac{[\mathrm{Tr}(K^{-1} \frac{\partial K}{\partial \Delta_1} K^{-1} \frac{\partial K}{\partial \Delta_2})]^2}{\mathrm{Tr}[(K^{-1} \frac{\partial K}{\partial \Delta_1})^2]\mathrm{Tr}[(K^{-1} \frac{\partial K}{\partial \Delta_2})^2]} \tag{140}$$

The first factor in the denominator is given by Eq. (G-20). Since the problem of estimating Δ_2 with Δ_1 known is precisely the same as that of estimating Δ_1 with Δ_2 known, the second factor in the denominator of Eq. (140) is also given by (G-20), a fact readily confirmed by direct computation. There remains only the calculation of the numerator of Eq. (140). The only new feature in it is the factor $\partial K/\partial \Delta_2$. The elements of this matrix can be inferred immediately from Eqs. (135)-(137), with $\tau_1 = \tau_2 = 0$

$$\frac{\partial}{\partial \Delta_2} E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = 0 \tag{141}$$

$$\frac{\partial}{\partial \Delta_2} E\{X_1(\omega_n)X_3^*(\omega_\ell)\} = \begin{cases} 0 & \ell = n+h \\ -\frac{T^2}{2} \frac{(-1)^{\ell-n-h}}{\ell-n-h} S[\frac{\pi}{T}(\ell+n-h)-\omega_D] & \ell \neq n+h \end{cases} \tag{142}$$

$$\frac{\partial}{\partial \Delta_2} E\{X_2(\omega_n)X_3^*(\omega_\ell)\} = \begin{cases} 0 & \ell = n+h-k \\ -\frac{T^2}{2} \frac{(-1)^{\ell-n-h+k}}{\ell-n-h+k} S[\frac{\pi}{T}(\ell+n-h+k)-\omega_D-\Delta_1] & \ell \neq n+h-k \end{cases} \tag{143}$$

The calculation can now proceed as in Appendix G. Details are given in Appendix H. According to Eq. (H-9) the coupling coefficient ρ has a numerical value of (1/4) completely independent of the spectral properties of the signal. The mean square estimation error for Δ_1 with Δ_2 unknown (or Δ_2 with Δ_1 unknown) is now easily written down from Eqs. (139) and (H-9):

$$D^2(\hat{\Delta}_1)\Big|_{\Delta_2 \text{ unknown}} = \frac{4/3}{\dfrac{T^3}{6\pi}\displaystyle\int_0^\infty \frac{S^2(\omega-\omega_D)}{3N_0\, S(\omega-\omega_D) + N_0^{\,2}}\, d\omega} \tag{144}$$

It is interesting to compare Eq. (144) with Eq. (98), the latter describing the mean square estimation error of the differential Doppler shift between the signals received at one pair of subarrays. When the signal to noise ratio in the signal band is large, so that the $N_0^{\,2}$ term in the integrand can be ignored, Eqs. (144) and (98) are identical. Evidently it makes no difference whether one uses the subarrays pairwise to obtain separate estimates of Δ_1 and Δ_2 or whether one works with all three subarray outputs simultaneously. When the signal to noise ratio is low even in the signal band the two estimation errors have the ratio

$$\frac{D^2(\Delta\hat{\omega})}{D^2(\hat{\Delta}_1)\Big|_{\Delta_2 \text{ unknown}}} = 1.5 \tag{145}$$

A modest gain can therefore be made in principle by processing the three subarray outputs simultaneously. It appears questionable, however, whether this relatively small gain would justify the increased complexity of instrumentation.

Appendix A. Derivation of $R_{ij}(t,\sigma)$.

From Eqs. (69), (70) and (71)

$$R_{12}(t,\sigma) = \sum_{q=1}^{N} \sum_{r=1}^{N} c_q c_r \; E\{\cos[(\omega_q+\omega_D)t-\phi_q]\cos[(\omega_r+\omega_D+\Delta\omega)(\sigma-\tau_0)-\phi_r]\}$$

$$= \sum_{q=1}^{N} \sum_{r=1}^{N} \frac{c_q c_r}{2}\Big(E\{\cos[(\omega_q+\omega_D)t+(\omega_r+\omega_D+\Delta\omega)\sigma-(\omega_r+\omega_D+\Delta\omega)\tau_0-\phi_q-\phi_r]\}$$

$$+ E\{\cos[(\omega_q+\omega_D)t-(\omega_r+\omega_D+\Delta\omega)\sigma+(\omega_r+\omega_D+\Delta\omega)\tau_0-\phi_q+\phi_r]\}\Big) \qquad \text{(A-1)}$$

Since the ϕ's are all statistically independent and uniformly distributed over $(0,2\pi)$, the only non-zero contribution is made by the second term in (A-1) which is non-zero for q=r. For that combination the ϕ's vanish and the averaging operation becomes trivial. Hence,

$$R_{12}(t,\sigma) = \sum_{q=1}^{N} \frac{c_q^2}{2} \cos[(\omega_q+\omega_D)(t-\sigma)-\Delta\omega\sigma+(\omega_q+\omega_D+\Delta\omega)\tau_0] \qquad \text{(A-2)}$$

Using Eq. (72) and recognizing that the time interval T_0 can be made arbitrarily large (so that $\Delta\omega \to 0$) one can convert the sum into an integral

$$R_{12}(t,\sigma) = \int_0^\infty d\omega \; S(\omega)\cos[(\omega+\omega_D)(t-\sigma)-\Delta\omega\sigma+(\omega+\omega_D+\Delta\omega)\tau_0] \qquad \text{(A-3)}$$

In completely analogous fashion

$$R_{11}(t,\sigma) = \int_0^\infty d\omega[S(\omega)+N_1(\omega)]\cos[(\omega+\omega_D)(t-\sigma)] \qquad \text{(A-4)}$$

$$R_{22}(t,\sigma) = \int_0^\infty d\omega[S(\omega)+N_2(\omega)]\cos[(\omega+\omega_D+\Delta\omega)(t-\sigma)] \qquad \text{(A-5)}$$

Note that the autocorrelation functions (A-4) and (A-5) incorporate the received noise spectra. R_{12} depends only on the signal because of the postulated independence of the noise at the two subarrays.

Appendix B. The elements of K.

Substituting Eq. (75) into Eq. (68) and using the exponential form of the cosine function:

$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} =$$

$$\frac{1}{2}\int_0^\infty d\omega S(\omega)e^{j(\omega+\omega_D+\Delta\omega)\tau_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} dte^{j(\omega+\omega_D-\omega_n)t}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\sigma e^{-j(\omega+\omega_D+\Delta\omega-\omega_\ell)\sigma}$$

$$+\frac{1}{2}\int_0^\infty d\omega S(\omega)e^{-j(\omega+\omega_D+\Delta\omega)\tau_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} dte^{-j(\omega+\omega_D+\omega_n)t}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\sigma e^{j(\omega+\omega_D+\Delta\omega+\omega_\ell)\sigma} \quad \text{(B-1)}$$

The t and σ integrations are easily performed

$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} =$$

$$\frac{T^2}{2}\int_0^\infty d\omega S(\omega)e^{j(\omega+\omega_D+\Delta\omega)\tau_0}\ \frac{\sin(\omega+\omega_D-\omega_n)\frac{T}{2}}{(\omega+\omega_D-\omega_n)\frac{T}{2}}\ \frac{\sin(\omega+\omega_D+\Delta\omega-\omega_\ell)\frac{T}{2}}{(\omega+\omega_D+\Delta\omega-\omega_\ell)\frac{T}{2}}$$

$$+\ \frac{T^2}{2}\int_0^\infty d\omega S(\omega)e^{-j(\omega+\omega_D+\Delta\omega)\tau_0}\ \frac{\sin(\omega+\omega_D+\omega_n)\frac{T}{2}}{(\omega+\omega_D+\omega_n)\frac{T}{2}}\ \frac{\sin(\omega+\omega_D+\Delta\omega+\omega_\ell)\frac{T}{2}}{(\omega+\omega_D+\Delta\omega+\omega_\ell)\frac{T}{2}} \quad \text{(B-2)}$$

Our signal spectrum $S(\omega)$ is concentrated in the neighborhood of $\omega = \omega_0$. The second integral in (B-2) is therefore essentially zero unless ω_n is near $-\omega_D$. Since we are only working with positive frequency Fourier coefficients this cannot occur and we can concentrate our attention on the first integral.

Define

$$\phi_{n\ell}(\omega) = \frac{\sin(\omega+\omega_D-\omega_n)\frac{T}{2}}{(\omega+\omega_D-\omega_n)\frac{T}{2}} \; \frac{\sin(\omega+\omega_D+\Delta\omega-\omega_\ell)\frac{T}{2}}{(\omega+\omega_D+\Delta\omega-\omega_\ell)\frac{T}{2}} \tag{B-3}$$

Then Eq. (B-2) becomes

$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = \frac{T^2}{2} e^{j(\omega_D+\Delta\omega)\tau_0} \int_{-\infty}^{\infty} d\omega e^{j\omega\tau_0} S(\omega)\phi_{n\ell}(\omega) \tag{B-4}$$

We have extended the lower limit of integration to $-\infty$ because the spectral function $S(\omega)$ was defined as zero for $\omega < 0$.

Next, using $F(\ )$ to designate the Fourier transform of the bracketed quantity, we obtain from Parseval's theorem

$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = \frac{T^2}{2} e^{j(\omega_D+\Delta\omega)\tau_0} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\gamma \, F(e^{j\omega\tau_0}S(\omega))F(\phi_{n\ell}(\omega))^* \tag{B-5}$$

Now

$$F\{e^{j\omega\tau_0}S(\omega)\} = \int_{-\infty}^{\infty} S(\omega)e^{-j\omega(\gamma-\tau_0)} \, d\omega = R_1(\gamma-\tau_0) \tag{B-6}$$

where

$$R_1(\gamma) = \int_{-\infty}^{\infty} S(\omega)e^{-j\omega\gamma} \, d\omega \tag{B-7}$$

Also

$$F\{\phi_{n\ell}(\omega)\} \equiv \Phi_{n\ell}(\gamma) = F\{\frac{\sin(\omega+\omega_D-\omega_n)\frac{T}{2}}{(\omega+\omega_D-\omega_n)\frac{T}{2}} \cdot \frac{\sin(\omega+\omega_D+\Delta\omega-\omega_\ell)\frac{T}{2}}{(\omega+\omega_D+\Delta\omega-\omega_\ell)\frac{T}{2}}\}$$

$$= \frac{1}{2\pi}\{ F[\frac{\sin(\omega+\omega_D-\omega_n)\frac{T}{2}}{(\omega+\omega_D-\omega_n)\frac{T}{2}}] \star F[\frac{\sin(\omega+\omega_D+\Delta\omega-\omega_\ell)\frac{T}{2}}{(\omega+\omega_D+\Delta\omega-\omega_\ell)\frac{T}{2}}]\} \tag{B-8}$$

The symbol $\star$ denotes convolution.

We know that

$$\tilde{F}\left[\frac{\sin(\omega+\omega_D-\omega_n)\frac{T}{2}}{(\omega+\omega_D-\omega_n)\frac{T}{2}}\right] = \frac{2\pi}{T}\, e^{j\gamma(\omega_D-\omega_n)}\ \mathrm{rect}(\gamma;\ \tfrac{T}{2}) \qquad \text{(B-9)}$$

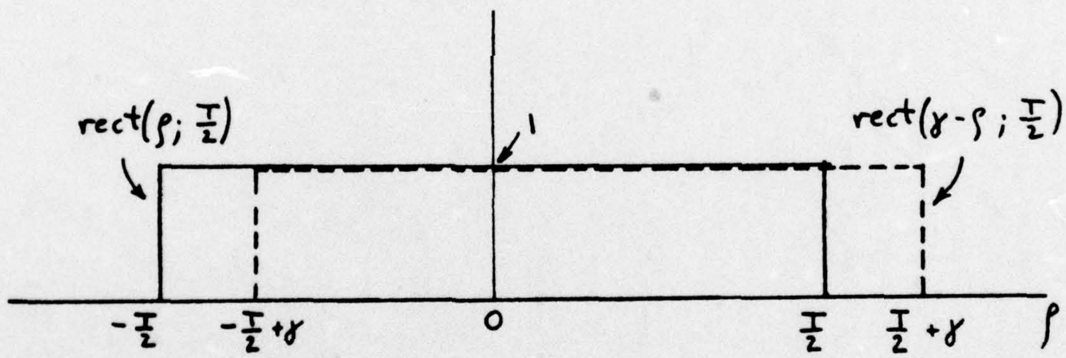
where $\mathrm{rect}(\gamma;\ \frac{T}{2})$ is the rectangle function defined by

$$\mathrm{rect}(\gamma;\ \tfrac{T}{2}) = \begin{cases} 1 & |\gamma| \le \frac{T}{2} \\ 0 & |\gamma| > \frac{T}{2} \end{cases} \qquad \text{(B-10)}$$

It follows that

$$\Phi_{n\ell}(\gamma) = \frac{2\pi}{T^2}\int_{-\infty}^{\infty} d\rho\, e^{j\rho(\omega_D-\omega_n)}\ \mathrm{rect}(\rho;\ \tfrac{T}{2})\, e^{j(\gamma-\rho)(\omega_D+\Delta\omega-\omega_\ell)}\ \mathrm{rect}(\gamma-\rho;\ \tfrac{T}{2}) \qquad \text{(B-11)}$$

The two rectangle functions are sketched below for $\gamma > 0$.



Hence, for $\gamma > 0$,

$$\Phi_{n\ell}(\gamma) = \frac{2\pi}{T^2}\, e^{j\gamma(\omega_D+\Delta\omega-\omega_\ell)} \int_{-\frac{T}{2}+\gamma}^{\frac{T}{2}} d\rho\, e^{j\rho(\omega_\ell-\omega_n-\Delta\omega)} \qquad \text{(B-12)}$$

After integration and a few steps of algebra

$$\Phi_{n\ell}(\gamma) = \frac{2\pi}{T}\, \frac{\sin[(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}(1-\frac{\gamma}{T})]}{(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}\ e^{j\gamma(\omega_D+\frac{\Delta\omega}{2}-\frac{\omega_\ell+\omega_n}{2})} \qquad \gamma > 0 \qquad \text{(B-13)}$$

Repeating the computation for $\gamma < 0$ one finds a change only in the sign of γ in the argument of the sinusoid. Thus for all γ

$$\Phi_{n\ell}(\gamma) = \frac{2\pi}{T}\,\frac{\sin[(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}(1-\frac{|\gamma|}{T})]}{(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}\,e^{j\gamma[\omega_D+\frac{1}{2}(\Delta\omega-\omega_\ell-\omega_n)]} \tag{B-14}$$

Now return to Eq. (B-5)

$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = \frac{T^2}{4\pi}\,e^{j(\omega_D+\Delta\omega)\tau_0}\int_{-\infty}^{\infty} d\gamma R_1(\gamma-\tau_0)\Phi_{n\ell}^*(\gamma) \tag{B-15}$$

$R_1(\gamma)$ is closely related to the signal autocorrelation. In fact, suppose that the signal spectrum is symmetrical about the center frequency $\omega = \omega_0$ so that

$$S(\omega) = G(\omega-\omega_0),\ G \text{ an even function.} \tag{B-16}$$

Then

$$R_1(\gamma) = \int_{-\infty}^{\infty} G(\omega-\omega_0)e^{-j\omega\gamma}d\omega$$

$$= e^{-j\omega_0\gamma}\int_{-\infty}^{\infty} G(x)\ e^{-x\gamma}dx = e^{-j\omega_0\gamma}R(\gamma) \tag{B-17}$$

where $R(\gamma)$ is the autocorrelation of the signal after a down-shift in frequency by ω_0 rad/sec. It follows that R_1 as well as R vanish for arguments in excess of the signal correlation time. For TW >> 1 this time is much smaller than T. Furthermore, the relative delay τ_0 must be much smaller than T if the two subarrays are to process information coherently. Under these conditions $\gamma << T$ throughout the effective range of integration of (B-15). Hence the sinusoidal term in (B-14) becomes effectively γ independent and one can write

$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} =$$

$$\frac{T}{2}\,\frac{\sin(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}{(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}\; e^{j(\omega_D+\Delta\omega)\tau_0}\int_{-\infty}^{\infty} d\gamma R_1(\gamma-\tau_0)e^{-j\gamma[\omega_D+\frac{1}{2}(\Delta\omega-\omega_\ell-\omega_n)]} \qquad \text{(B-18)}$$

The integral is simply a Fourier transform of R_1 and the result is therefore easily stated in terms of $S(\omega)$. After a few steps of algebra

$$E\{X_1(\omega_n)X_2^*(\omega_\ell)\} =$$

$$\pi T\,\frac{\sin(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}{(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}\; S\left(\frac{\omega_\ell+\omega_n-\Delta\omega}{2}-\omega_D\right)e^{j\frac{\tau_0}{2}(\omega_n+\omega_\ell+\Delta\omega)} \qquad \text{(B-19)}$$

The remaining terms of the data covariance matrix are almost obvious by inspection. When both Fourier coefficients come from the same subarray output there are no differential delays or differential Doppler shifts. The signal spectrum is shifted by ω_D at the first subarray, by $\omega_D + \Delta\omega$ at the second. Correlation now exists only when the two indices are the same. When they are the same, however, there is correlation for the noise as well as the signal. The correct form of the covariance elements is now evident from Eq. (B-19).

$$E\{X_1(\omega_n)X_1^*(\omega_\ell)\} = \begin{cases} \pi T[S(\omega_n-\omega_D)+N_1(\omega_n)] & n = \ell \\ 0 & n \neq \ell \end{cases} \qquad \text{(B-20)}$$

$$E\{X_2(\omega_n)X_2^*(\omega_\ell)\} = \begin{cases} \pi T[S(\omega_n-\omega_D-\Delta\omega)+N_2(\omega_n)] & n = \ell \\ 0 & n \neq \ell \end{cases} \qquad \text{(B-21)}$$

Appendix C. Computation of K^{-1}.

Rearrange the data vector as follows

$$\underline{x} = X_1(\omega_1), X_2(\omega_{1+k}), X_1(\omega_2) X_2(\omega_{2+k}), \ldots X_1(\omega_{N-k}), X_2(\omega_N) \,|\, X_1(\omega_{N-k+1}) \ldots X_1(\omega_N) \,|\, X_2(\omega_1) \ldots X_2(\omega_k) \,| \tag{C-1}$$

In this arrangement the first 2(N-k) terms pair all indices from subarrays 1 and 2 for which there is non-zero correlation according to Eq. (83). The remaining terms are Fourier coefficients near the upper edge of the processed frequency band of subarray 1 and the lower edge of the processed frequency band of subarray 2. Terms in the latter two categories are uncorrelated with all other elements of the data vector.

The data covariance matrix now assumed the block diagonal form

$$K = \begin{array}{c} \begin{array}{ccc} \phantom{\begin{bmatrix} A_1 & & & \\ \end{bmatrix}} & k & k \end{array} \\ \left[\begin{array}{cccc:c:c} [A_1] & & & & & \\ & [A_2] & & 0 & & \\ & & \ddots & & 0 & 0 \\ & 0 & & & & \\ & & & [A_{N-k}] & & \\ \hdashline & 0 & & & B & 0 \\ \hdashline & 0 & & & 0 & C \end{array}\right] \end{array} \tag{C-2}$$

From Eqs. (78),(79) and (83) the 2 x 2 matrices A_n are given by

$$A_n = \pi T \left[\begin{array}{c|c} S(\omega_n-\omega_D)+N_1(\omega_n) & S(\omega_n-\omega_D)e^{j\tau_0(\omega_n+\Delta\omega)} \\ \hline S(\omega_n-\omega_D)e^{-j\tau_0(\omega_n+\Delta\omega)} & S(\omega_n-\omega_D)+N_2(\omega_n+\Delta\omega) \end{array}\right] \tag{C-3}$$

To simplify the algebra we shall assume that $N_1(\omega) = N_2(\omega) = N_0$, a constant. Thus we are postulating that the noise at each subarray is spectrally flat over the processed frequency band and of the same power level at each subarray. Since the signal has a very narrow bandwidth and any Doppler shifts will be small compared with the center frequency of the signal this assumption does not appear to impose any serious limitations.

The inversion of A_n is now trivial

$$A_n^{-1} = \frac{1}{\pi T}\,\frac{1}{2N_0S(\omega_n-\omega_D)+N_0^2}\left[\begin{array}{c|c} S(\omega_n-\omega_D)+N_0 & -S(\omega_n-\omega_D)e^{j(\omega_n+\Delta\omega)\tau_0} \\ \hline -S(\omega_n-\omega_D)e^{-j(\omega_n+\Delta\omega)\tau_0} & S(\omega_n-\omega_D)+N_0 \end{array}\right] \tag{C-4}$$

The matrices B and C are diagonal. The frequencies contained in B are all within $\Delta\omega$ of the upper end of the processed band, those in C within $\Delta\omega$ of the lower end. In practice one would certainly process a band wide enough so that no significant signal components are lost under any conceivable Doppler

shifts. Hence no signal power should lie with $\Delta\omega$ of the band edge and we have

$$B = C = \pi T N_0 I \tag{C-5}$$

where I is the $k \times k$ identity matrix. The inverse of Eq. (C-2) is therefore

$$K^{-1} = \begin{array}{c} \begin{array}{cc} \qquad 2(N-K) \qquad\qquad & \qquad 2K \end{array} \\ \left[\begin{array}{cccc|ccc} [A_1^{-1}] & & 0 & & & & \\ & [A_2^{-1}] & & & & 0 & \\ & & \ddots & & & & \\ 0 & & & [A_{N-K}^{-1}] & & & \\ \hline & & & & \frac{1}{\pi T N_0} & & 0 \\ & 0 & & & & \ddots & \\ & & & & 0 & & \frac{1}{\pi T N_0} \end{array}\right] \end{array} \tag{C-6}$$

Finally, rearranging the data vector back to the form of Eq. (84) [and hence K to the form of Eq. (85)] we obtain the result shown in Eq. (86).

Appendix D. Computation of $\frac{dK}{d(\Delta\omega)}$.

From Eq. (77)

$$\frac{d}{d(\Delta\omega)}\,E\{X_1(\omega_n)X_2^*(\omega_\ell)\} =$$

$$\pi T\{\frac{-(\omega_\ell-\omega_n-\Delta\omega)\frac{T^2}{4}\;\cos(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}+\frac{T}{2}\sin(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}{(\omega_\ell-\omega_n-\Delta\omega)^2\;\frac{T^2}{4}}\;S(\frac{\omega_\ell+\omega_n-\Delta\omega}{2}-\omega_D)e^{j\frac{\tau_0}{2}(\omega_n+\omega_\ell+\Delta\omega)}$$

$$+\frac{\sin(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}{(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2}}[-\frac{1}{2}\,S'(\frac{\omega_\ell+\omega_n-\Delta\omega}{2}-\omega_D)+j\frac{\tau_0}{2}\,S(\frac{\omega_\ell+\omega_n-\Delta\omega}{2}-\omega_D)]e^{j\frac{\tau_0}{2}(\omega_n+\omega_\ell+\Delta\omega)}\} \tag{D-1}$$

Now write $\Delta\omega = \frac{2\pi k}{T}$ and note that with this choice $(\omega_\ell-\omega_n-\Delta\omega)\frac{T}{2} = (\ell-n-k)\pi$. It follows immediately that the first line of Eq. (D-1) vanishes when $\ell-n-k = 0$ whereas the second line vanishes whenever $\ell-n-k \neq 0$. Thus Eq. (D-1) reduces to the relatively simple expression

$$\frac{d}{d(\Delta\omega)}E\{X_1(\omega_n)X_2^*(\omega_\ell)\} = \begin{cases} \frac{\pi T}{2}[j\tau_0\;S(\omega_n-\omega_D)-S'(\omega_n-\omega_D)]e^{j\tau_0(\omega_n+\Delta\omega)} & \ell = n+k \\ -\frac{T^2}{2}\;\frac{(-1)^{\ell-n-k}}{\ell-n-k}\;S[\frac{\pi}{T}(\ell+n-k)-\omega_D]e^{j\frac{\tau_0}{2}\pi(n+\ell+k)} & \ell \neq n+k \end{cases} \tag{D-2}$$

Appendix E. Computation of $\mathrm{Tr}\{(K^{-1}\frac{dK}{d(\Delta\omega)})^2\}$

The matrix K^{-1} is given by Eq. (86). For greater ease in computation we regard it as composed of the blocks shown in Eq. (E-1).

$$K^{-1} = \frac{1}{\pi T}\begin{array}{cc} \begin{array}{cccc} \overbrace{\qquad\quad}^{N-k} & \overbrace{\quad}^{k} & \overbrace{\quad}^{k} & \overbrace{\qquad\quad}^{N-k} \end{array} & \\ \left[\begin{array}{c|c|c|c} B & 0 & 0 & -D \\ \hline 0 & N & 0 & 0 \\ \hline 0 & 0 & N & 0 \\ \hline -D^* & 0 & 0 & B \end{array}\right] & \begin{array}{l} \}\,N-k \\ \}\,k \\ \}\,k \\ \}\,N-k \end{array} \end{array}$$

(E-1)[1]

Here

$$B = \underbrace{\begin{bmatrix} B_1 & & & & 0 \\ & B_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & B_{N-k} \end{bmatrix}}_{N-K} \Bigg\}\, N-K \qquad \text{(E-2)}$$

---

[1]The arrangement implies that $k \geq 0$. This is insured by a convention which applies the label 2 to the subarray receiving the signal with the higher center frequency.

$$D = \overbrace{\begin{bmatrix} D_1 e^{j\rho_1} & & & & 0 \\ & D_2 e^{j\rho_1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & D_{N-k} e^{j\rho_{N-k}} \end{bmatrix}}^{N-k} \left.\vphantom{\begin{bmatrix} 1\\1\\1\\1\\1 \end{bmatrix}}\right\} N-k \qquad \text{(E-3)}$$

and

$$N = \frac{1}{N_0} \overbrace{\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \cdot & \\ 0 & & & 1 \end{bmatrix}}^{k} \left.\vphantom{\begin{bmatrix} 1\\1\\1\\1 \end{bmatrix}}\right\} k \qquad \text{(E-4)}$$

All of these matrices are diagonal, B and D of dimension (N-k)x(N-k), N of dimension k x k. The blocks labelled 0 in Eq. (E-1) have only zero entries.

Next we represent the matrix $\frac{dK}{d(\Delta\omega)}$ by a similar arrangement of blocks. Using Eqs. (93)-(95)

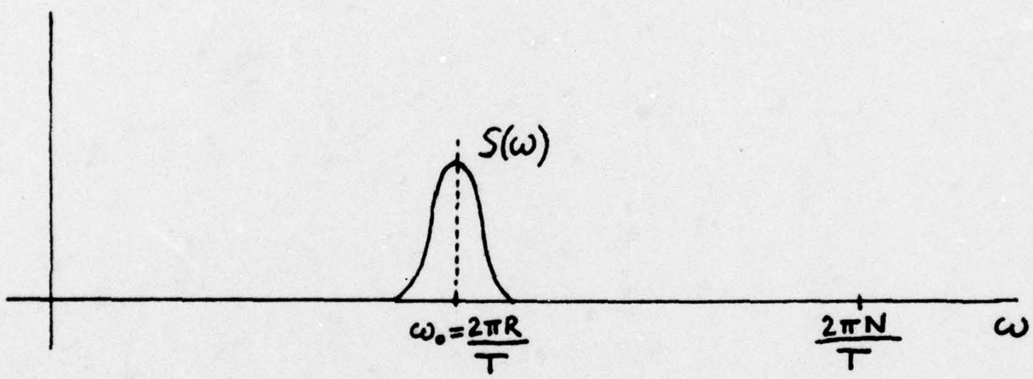
$$\frac{dK}{d(\Delta\omega)} = \frac{T^2}{2} \begin{array}{c} \begin{array}{cccc} N-k & k & k & N-k \end{array} \\ \left[\begin{array}{c:c:c:c} 0 & 0 & F & A \\ \hdashline 0 & 0 & H & G \\ \hdashline F^* & H^* & 0 & 0 \\ \hdashline A^* & G^* & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \} N-k \\ \} k \\ \} k \\ \} N-k \end{array} \qquad \text{(E-5)}$$

The blocks labelled F, G, and H have the following properties:

1) In F and H the ℓ index is between 1 and k

2) In G and H the n index is between N-k and N

We are dealing with a narrowband signal concentrated near ω_0. [See figure.] The Doppler shift ω_D and differential Doppler shift $\Delta\omega = 2\pi k/T$ are certainly small compared with ω_0. Let $\omega_0 = 2\pi R/T$, so that R is the index of



of the Fourier coefficient associated with the center frequency of the signal. Hence the factor $S[\frac{\pi}{T}(\ell+n-k)-\omega_D]$ in Eq. (93) is different from zero only when $\ell+n-k$ is near 2R.

In F and H, according to (1), ℓ is between 1 and k. It follows that n is of the order 2R (compared to which k is negligible). Thus the factor $\ell-n-k$ in the denominator of Eq. (93) has a value close to -2R. From the definition of R we have

$$2R = \frac{\omega_0 T}{\pi} = 2\,\frac{T}{T_0} \tag{E-6}$$

where T_0 is the period of the center frequency. In all interesting situations the ratio T/T_0 should be extremely large and as a consequence all elements of F and H are essentially zero.

A very similar argument applies to G. Here, according to (2), n is near N so that ℓ must be near 2R-N if there is to be a non-zero contribution. This says that ℓ-n-k must be close to 2R-2N. N characterizes the upper edge of the processed frequency band and will certainly be very much larger than R. N can, in fact, be arbitrarily large so that the elements of G are arbitrarily close to zero.

Eq. (E-5) can now be simplified to the form

$$\frac{dK}{d(\Delta\omega)} = \frac{T^2}{2}\; \begin{array}{cc} \begin{array}{cccc} \overbrace{\qquad\quad}^{N-k} & \overbrace{\quad}^{k} & \overbrace{\quad}^{k} & \overbrace{\qquad\quad}^{N-k} \end{array} & \\ \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & A \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline A^* & 0 & 0 & 0 \end{array}\right] & \begin{array}{l} \}\,N-k \\ \}\,k \\ \}\,k \\ \}\,N-k \end{array} \end{array} \tag{E-7}$$

The elements a_{nm} of A are given by Eq. (93) with $\ell-k \equiv m$. Thus

$$a_{nm} = \begin{cases} 0 & m = n \\ -\dfrac{(-1)^{m-n}}{m-n}\, S[\frac{\pi}{T}(m+n)-\omega_D]\, e^{j\pi\frac{\tau_0}{T}(n+m+2k)} & m \neq n \end{cases} \tag*{(E-8)[1]}$$

---

[1]The factor $T^2/2$ in Eq. (93) has been removed from the definition of A by its explicit appearance in Eq. (E-7).

Straightforward matrix multiplication now yields from Eqs. (E-1) and (E-7)

$$K^{-1}\frac{dK}{d(\Delta\omega)} = \frac{T}{2\pi}\begin{array}{cc} \begin{array}{cccc} \overbrace{\qquad}^{N-k} & \overbrace{\quad}^{k} & \overbrace{\quad}^{k} & \overbrace{\qquad}^{N-k} \end{array} & \\ \left[\begin{array}{c|c|c|c} -DA^* & 0 & 0 & BA \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline BA^* & 0 & 0 & -D^*A \end{array}\right] & \begin{array}{l} \}\,N-k \\ \}\,k \\ \}\,k \\ \}\,N-k \end{array} \end{array} \tag{E-9}$$

Therefore

$$\left(K^{-1}\frac{dK}{d(\Delta\omega)}\right)^2 = \frac{T^2}{4\pi^2}\begin{array}{cc} \begin{array}{cccc} \overbrace{\qquad}^{N-k} & \overbrace{\quad}^{k} & \overbrace{\quad}^{k} & \overbrace{\qquad}^{N-k} \end{array} & \\ \left[\begin{array}{c|c|c|c} \begin{array}{c} DA^*DA^* \\ + \\ BABA^* \end{array} & & & \\ \hline & 0 & & \\ \hline & & 0 & \\ \hline & & & \begin{array}{c} BA^*BA \\ + \\ D^*AD^*A \end{array} \end{array}\right] & \begin{array}{l} \}\,N-k \\ \}\,k \\ \}\,k \\ \}\,N-k \end{array} \end{array} \tag{E-10}$$

Only the blocks along the principal diagonal have been filled in since we are only interested in the trace of the matrix.

The required trace is now easily computed

$$\mathrm{Tr}\left\{\left(K^{-1}\frac{dK}{d(\Delta\omega)}\right)^2\right\} = \frac{T^2}{4\pi^2}\left\{\mathrm{Tr}(DA^*DA^*) + \mathrm{Tr}(BABA^*) + \mathrm{Tr}(BA^*BA) + \mathrm{Tr}(D^*AD^*A)\right\} \tag{E-11}$$

Using Eqs. (E-3), (E-8) and (89)

$$\mathrm{Tr}(DA^*DA^*) = \sum_{n=1}^{N-k} \sum_{m=1}^{N-k} D_n e^{j\rho_n} D_m e^{j\rho_m} a^*_{mn} a^*_{nm}$$

$$= \sum_{n} \sum_{m \neq n} D_n D_m e^{j2\pi \frac{\tau_0}{T}(n+m+2k)} \frac{1}{-(m-n)^2} S^2[\frac{\pi}{T}(m+n)-\omega_D] e^{-j2\pi\frac{\tau_0}{T}(n+m+2k)}$$

$$= -\sum_{n} \sum_{m \neq n} \frac{1}{(m-n)^2} D_n D_m S^2[\frac{\pi}{T}(m+n)-\omega_D] \tag{E-12}$$

Similarly

$$\mathrm{Tr}(D^*AD^*A) = -\sum_{n} \sum_{m \neq n} \frac{1}{(m-n)^2} D_n D_m S^2[\frac{\pi}{T}(m+n)-\omega_D] \tag{E-13}$$

Also from Eqs. (E-2) and (E-8)

$$\mathrm{Tr}(BABA^*) = \mathrm{Tr}(BA^*BA) = \sum_{n=1}^{k-1} \sum_{m=1}^{k-1} B_n B_m a_{nm} a^*_{nm} = \sum_{n} \sum_{m \neq n} \frac{1}{(m-n)^2} B_n B_m S^2[\frac{\pi}{T}(n+m)-\omega_D] \tag{E-14}$$

Hence

$$\mathrm{Tr}\{(K^{-1} \frac{dK}{d(\Delta\omega)})^2\} = \frac{T^2}{2\pi^2} \sum_{n} \sum_{m \neq n} \frac{1}{(m-n)^2} [B_n B_m - D_n D_m] S^2[\frac{\pi}{T}(n+m)-\omega_D] \tag{E-15}$$

From the definitions of B_n and D_n [Eqs. (87) and (88)]

$$B_n B_m - D_n D_m = \frac{N_0[S(\omega_n-\omega_D) + S(\omega_m-\omega_D)] + N_0^2}{[2N_0 S(\omega_n-\omega_D) + N_0^2][2N_0 S(\omega_m-\omega_D) + N_0^2]} \tag{E-16}$$

Make the change of index

$$m - n = r \tag{E-17}$$

Then Eq. (E-15) assumes the form

$$\mathrm{Tr}\{(K^{-1}\frac{dK}{d(\Delta\omega)})^2\} =$$

$$\frac{T^2}{2\pi^2}\sum_n\sum_{r\neq 0}\frac{1}{r^2}\frac{N_0[S(\omega_n-\omega_D)+S(\omega_n+\frac{2\pi r}{T}-\omega_D)]+N_0^{\,2}}{[2N_0S(\omega_n-\omega_D)+N_0^{\,2}][2N_0S(\omega_n+\frac{2\pi r}{T}-\omega_D)+N_0^{\,2}]}\,S^2(\omega_n+\frac{\pi r}{T}-\omega_D) \tag{E-18}$$

The n index formally runs from 1 to N-k. If the arbitrary upper processing limit N is taken sufficiently large so that all signal power lies below $\frac{2\pi}{T}(N-k)$, the n sum can be extended to infinity.

For large TW products and smoothly varying power spectra, $S(\omega_n)$ changes only insignificantly over any frequency increment $\Delta\omega = 2\pi/T$. The n sum can then be converted into an integral

$$\mathrm{Tr}\{(K^{-1}\frac{dK}{d(\Delta\omega)})^2\} =$$

$$\frac{T^3}{4\pi^3}\sum_{r\neq 0}\frac{1}{r^2}\int_0^\infty\frac{N_0[S(\omega'-\omega_D)+S(\omega'+\frac{2\pi r}{T}-\omega_D)]+N_0^{\,2}}{[2N_0S(\omega'-\omega_D)+N_0^{\,2}][2N_0S(\omega'+\frac{2\pi r}{T}-\omega_D)+N_0^{\,2}]}\,S^2(\omega'+\frac{\pi r}{T}-\omega_D)\,d\omega' \tag{E-19}$$

Because of the rapidly varying factor $1/r^2$ only a few terms of the r sum contribute significantly to the result.

A somewhat more symmetrical form can be obtained with the change of variable

$$\omega' + \frac{\pi r}{T} = \omega \tag{E-20}$$

Since there is no signal power near $\omega = 0$ the lower limit can remain zero and Eq. (E-19) becomes

$$\mathrm{Tr}\{(K^{-1}\frac{dK}{d(\Delta\omega)})^2\} =$$

$$\frac{T^3}{4\pi^3}\sum_{r\neq 0}\frac{1}{r^2}\int_0^\infty \frac{N_0[S(\omega-\frac{\pi r}{T}-\omega_D)+S(\omega+\frac{\pi r}{T}-\omega_D)]+N_0^2}{[2N_0S(\omega-\frac{\pi r}{T}-\omega_D)+N_0^2][2N_0S(\omega+\frac{\pi r}{T}-\omega_D)+N_0^2]}S^2(\omega-\omega_D)\,d\omega \qquad \text{(E-21)}$$

Appendix F. Computation of $K^{-1}\frac{\partial K}{\partial \alpha}$

Eq. (E-1) shows K^{-1} decomposed into the square diagonal matrices B,D,N and several rectangular null matrices. A similar representation of $\frac{\partial K}{\partial \alpha}$ is according to Eq. (85)

$$\frac{\partial K}{\partial \alpha} = \begin{array}{cc} \begin{array}{cccc} N-k & k & k & N-k \end{array} & \\ \left[\begin{array}{c:c:c:c} P & 0 & 0 & R \\ \hdashline 0 & S & 0 & 0 \\ \hdashline 0 & 0 & T & 0 \\ \hdashline R & 0 & 0 & Q \end{array}\right] & \begin{array}{l} \}N-k \\ \}k \\ \}k \\ \}N-k \end{array} \end{array} \tag{F-1}$$

The matrices P,Q,R,S, and T are all square and diagonal.

Multiplication of Eqs. (E-1) and (F-1) yields

$$K^{-1}\frac{\partial K}{\partial \alpha} = \frac{1}{\pi T} \begin{array}{cc} \begin{array}{cccc} N-k & k & k & N-k \end{array} & \\ \left[\begin{array}{c:c:c:c} BP-D^*R & 0 & 0 & BR-DQ \\ \hdashline 0 & NS & 0 & 0 \\ \hdashline 0 & & NT & 0 \\ \hdashline -D^*P+BR & 0 & 0 & -D^*R+BQ \end{array}\right] & \begin{array}{l} \}N-k \\ \}k \\ \}k \\ \}N-k \end{array} \end{array} \tag{F-2}$$

Since all of the matrices in F-2 are square and diagonal, non-zero elements occur only along the principal diagonal and the same pair of secondary diagonals as in K. Thus $K^{-1}\frac{\partial K}{\partial \alpha}$ has the form shown in Eq. (116).

Appendix G. Computation of $\mathrm{Tr}\{(K^{-1}\frac{\partial K}{\partial \Delta_1})^2\}$ for 3 subarrays.

Rearrange the data vector as follows (let $h \geq k \geq 0$)

$$
\begin{aligned}
\underline{x} = {} & X_1(\omega_1), X_2(\omega_{1+k}), X_3(\omega_{1+h}) \ldots X_1(\omega_{N-h}), X_2(\omega_{N-h+k}), X_3(\omega_N) \;\vdots \\
& X_1(\omega_{N-k+1}), X_2(\omega_{N-h+k+1}), \ldots X_1(\omega_{N-k}), X_2(\omega_N), \;\vdots\; X_1(\omega_{N-k+1}), \ldots X_1(\omega_N), \;\vdots \\
& X_2(\omega_1), X_3(\omega_{1+h-k}), \ldots X_2(\omega_k), X_3(\omega_h), \;\vdots\; X_3(\omega_1), \ldots X_3(\omega_{h-k}) \;\vdots
\end{aligned}
\tag{G-1}
$$

All frequencies in the second and third lines of this expression are within Δ_2 of either zero or ω_N. As in Appendix C we shall assume that the signal band is sufficiently narrow and ω_N has been chosen sufficiently large so that there is no appreciable signal power in these border regions. Then the last two lines contain only noise (uncorrelated from sensor to sensor) and the data covariance assumes the rather simple form

$$
K = \left[\begin{array}{cccc|ccc}
[M_1] & & & & & & \\
 & [M_2] & & & & 0 & \\
 & & \ddots & & & & \\
 & & & [M_{N-h}] & & & \\
\hline
 & & & & \frac{1}{N_0} & & \\
 & 0 & & & & \ddots & \\
 & & & & & & \frac{1}{N_0}
\end{array}\right]
\tag{G-2}
$$

The M_n are 3 x 3 matrices readily obtainable from Eqs. (126)-(128) and (131)-(133) [with $\tau_1 = \tau_2 = 0$].

$$M_n = \pi T \left[\begin{array}{c|c|c} S(\omega_n - \omega_D) + N_0 & S(\omega_n - \omega_D) & S(\omega_n - \omega_D) \\ \hline S(\omega_n - \omega_D) & S(\omega_n - \omega_D) + N_0 & S(\omega_n - \omega_D) \\ \hline S(\omega_n - \omega_D) & S(\omega_n - \omega_D) & S(\omega_n - \omega_D) + N_0 \end{array}\right] \qquad \text{(G-3)}$$

The inverse of M_n is easily computed

$$M_n^{-1} = \frac{1}{\pi T N_0 [3S(\omega_n - \omega_D) + N_0]} \left[\begin{array}{c|c|c} 2S(\omega_n - \omega_D) + N_0 & -S(\omega_n - \omega_D) & -S(\omega_n - \omega_D) \\ \hline -S(\omega_n - \omega_D) & 2S(\omega_n - \omega_D) + N_0 & -S(\omega_n - \omega_D) \\ \hline -S(\omega_n - \omega_D) & -S(\omega_n - \omega_D) & 2S(\omega_n - \omega_D) + N_0 \end{array}\right] \qquad \text{(G-4)}$$

Now rearranging the data vector back to its original form we obtain the following explicit version for the inverse of Eq. (134).

$$K^{-1} = \frac{1}{\pi T}\left[\begin{array}{c|c|c}
\begin{matrix} A_1 & & \\ & \ddots & \\ & & A_N \end{matrix} &
\begin{matrix} \overbrace{0}^{k} & \overbrace{\begin{matrix} -B_1 & & \\ & \ddots & \\ & & -B_{N-k} \end{matrix}}^{N-k} \\ 0 & 0 \end{matrix} &
\begin{matrix} \overbrace{0}^{h} & \overbrace{\begin{matrix} -B_1 & & \\ & \ddots & \\ & & -B_{N-k} \end{matrix}}^{N-h} \\ 0 & 0 \end{matrix} \\
\hline
\begin{matrix} 0 & 0 \\ \begin{matrix} -B_1 & & \\ & \ddots & \\ & & -B_{N-k} \end{matrix} & 0 \end{matrix} &
\begin{matrix} \frac{1}{N_0} & 0 \\ 0 & \begin{matrix} A_1 & & \\ & \ddots & \\ & & A_{N-k} \end{matrix} \end{matrix} &
\begin{matrix} 0 & 0 \\ 0 & \begin{matrix} -B_1 & & \\ & \ddots & \\ & & -B_{N-h} \end{matrix} \\ 0 & 0 \end{matrix} \\
\hline
\begin{matrix} 0 & 0 \\ \underbrace{\begin{matrix} -B_1 & & \\ & \ddots & \\ & & -B_{N-h} \end{matrix}}_{N-h} & \underbrace{0}_{h} \end{matrix} &
\begin{matrix} 0 & 0 \\ \underbrace{0}_{k} & \underbrace{\begin{matrix} -B_1 & & \\ & \ddots & \\ & & -B_{N-h} \end{matrix}}_{N-h} \end{matrix} &
\begin{matrix} \frac{1}{N_0} \cdots \frac{1}{N_0} & 0 \\ \underbrace{0}_{h} & \underbrace{\begin{matrix} A_1 & & \\ & \ddots & \\ & & A_{N-h} \end{matrix}}_{N-h} \end{matrix}
\end{array}\right]
\begin{array}{l} \}\,N-h \\ \}\,h \\ \}\,k \\ \}\,N-h \\ \}\,h-k \\ \}\,h \\ \}\,N-h \end{array}
\qquad (G-5)$$

Here

$$A_n = \frac{2S(\omega_n-\omega_D)+N_0}{N_0[3S(\omega_n-\omega_D)+N_0]} \qquad \text{(G-6)}$$

$$B_n = \frac{S(\omega_n-\omega_D)}{N_0[3S(\omega_n-\omega_D)+N_0]} \qquad \text{(G-7)}$$

All entries in Eq. (G-5) other than the indicated diagonals are zero.

The form of $\frac{\partial K}{\partial \Delta_1}$ is obtainable directly from Eqs. (135)-(137). As in the two subarray case we ignore the "edge effects" near the upper and lower end of the processed frequency band [see transition from Eq. (E-5) to (E-7)]. Then

$$\frac{\partial K}{\partial \Delta_1} = \frac{T^2}{2}\begin{bmatrix} 0 & C & 0 \\ C^* & 0 & D \\ 0 & D^* & 0 \end{bmatrix} \qquad \text{(G-8)}$$

C is an (N-k) X (N-k) matrix with the elements

$$c_{nm} = \begin{cases} 0 & m=n \\ -\dfrac{(-1)^{m-n}}{m-n} S[\frac{\pi}{T}(m+n)-\omega_D] & m \neq n \end{cases} \tag{G-9}$$

D is an (N-h) X (N-h) matrix with the elements

$$d_{nm} = \begin{cases} 0 & m=n \\ \dfrac{(-1)^{m-n}}{m-n} S[\frac{\pi}{T}(m+n)-\omega_D] & m \neq n \end{cases} \tag{G-10}$$

Eq. (G-5) can be put into a form similar to (G-8) with the definitions

$$A^{(L)} = \begin{bmatrix} A_1 & & & & \\ & A_2 & & O & \\ & & \ddots & & \\ & O & & \ddots & \\ & & & & A_{N-L} \end{bmatrix}$$

$$B^{(L)} = \begin{bmatrix} B_1 & & & & \\ & B_2 & & O & \\ & & \ddots & & \\ & O & & \ddots & \\ & & & & B_{N-L} \end{bmatrix} \tag{G-11}$$

Then

$$K^{-1} = \frac{1}{\pi T}\begin{bmatrix} A^{(k)} & 0 & 0 & -B^{(k)} & 0 & -B^{(h)} \\ 0 & 1/N_0 & & 0 & & 0 \\ 0 & & 1/N_0 & 0 & & 0 \\ -B^{(k)} & 0 & 0 & A^{(k)} & 0 & -B^{(h)} \\ & & & & & 0 \\ 0 & & & 0 & 1/N_0 \quad 1/N_0 & 0 \\ -B^{(h)} & 0 & 0 & -B^{(h)} & 0 \quad 0 & A^{(h)} \end{bmatrix} \tag{G-12}$$

The matrix product $K^{-1} \frac{\partial K}{\partial \Delta_1}$ is now easily computed.

$$K^{-1}\frac{\partial K}{\partial \Delta_1} = \frac{T}{2\pi} \qquad \begin{bmatrix} -B^{(k)}C^* & 0 & 0 & A^{(k)}C - B^{(h)}D^* & 0 & -B^{(h)}D \\ 0 & & & 0 & & 0 \\ 0 & & & 0 & & 0 \\ A^{(k)}C^* & 0 & 0 & -B^{(k)}C - B^{(h)}D^* & 0 & A^{(h)}D \\ & & & & & 0 \\ 0 & & & 0 & & 0 \\ -B^{(h)}C^{*(h)} & 0 & 0 & -B^{(h)}C^{(h)} + A^{(h)}D^* & 0 \quad 0 & -B^{(h)}D \end{bmatrix} \qquad \text{(G-13)}$$

In analogy with Eq. (G-11), $C^{(h)}$ is used to designate the matrix C with the last h-k rows and columns deleted. The second column of Eq. (G-13) has components of different dimensionality. This is to be interpreted as follows: Before addition to $A^{(k)}C$, the matrix $-B^{(h)}D$ is raised to dimension (N-k) X (N-k) by adding rows and columns of zeros at the bottom and at the right.

A straightforward computation now yields

$$\mathrm{Tr}\{K^{-1}\frac{\partial K}{\partial\Delta_1})^2\} =$$

$$\frac{T^2}{4\pi^2}\mathrm{Tr}\{B^{(k)}C^*B^{(k)}C^* + B^{(k)}CB^{(k)}C + 2A^{(k)}CA^{(k)}C^* - 2(B^{(h)}D^*)A^{(k)}C^*$$
$$+ 2B^{(k)}C(B^{(h)}D^*) + 2B^{(h)}DB^{(h)}C^{*(h)} - 2A^{(h)}DB^{(h)}C^{(h)}$$
$$+ 2A^{(h)}DA^{(h)}D^* + B^{(h)}DB^{(h)}D + B^{(h)}D^*B^{(h)}D^*\} \tag{G-14}$$

Apparently inconsistent matrix products such as $(B^{(h)}D)A^{(K)}C^*$ are made meaningful by adding rows and columns of zeros at the bottom and right of the (N-h) X (N-h) matrix.

The problem of different dimensionality in different components of (G-14) becomes trivial when the trace is written out as a sum. When the indices n and m are very different Eqs. (G-9) and (G-10) indicate that the terms in question are small. When both m and n are within h of the upper bound N they are also small because there is no signal power near the edge of the processed band. Thus both indices can be allowed to run over all integers and Eq. (G-14) becomes

$$\mathrm{Tr}\{(K^{-1}\frac{\partial K}{\partial\Delta_1})^2\} = \frac{T^2}{4\pi^2}\sum_n\sum_m\{2B_nB_mc_{mn}c_{nm} + A_nA_mc_{nm}c_{nm} - 2B_nA_md_{mn}c_{nm}$$
$$+ 2B_nB_mc_{nm}d_{nm} + 2B_nB_md_{nm}c_{nm} - 2A_nB_md_{nm}c_{mn} + 2A_nA_md_{nm}d_{nm}$$
$$+ 2B_nB_md_{nm}d_{mn}\} \tag{G-15}$$

According to Eqs. (G-9) and (G-10)

$$d_{nm} = -d_{mn} = -c_{nm} = c_{mn} \tag{G-16}$$

hence

$$\mathrm{Tr}\{K^{-1} \frac{\partial K}{\partial \Delta_1})^2\} = \frac{T^2}{2\pi^2} \sum_n \sum_m \{2A_nA_m - 4B_nB_m - B_nA_m - A_nB_m\}c_{nm}^2 \tag{G-17}$$

Now substituting the definitions (G-6),(G-7) and (G-9) and making the change of variables $m = n+r$

$$\mathrm{Tr}\{(K^{-1} \frac{\partial K}{\partial \Delta_1})^2\} =$$

$$\frac{T^2}{2\pi^2} \sum_{r\neq 0} \frac{1}{r^2} \sum_n \frac{3[S(\omega_n-\omega_D)+S(\omega_{n+r}-\omega_D)]+2N_0}{N_0[3S(\omega_n-\omega_D)+N_0][3S(\omega_{n+r}-\omega_D)+N_0]} S^2[\frac{\pi}{T}(2n+r)-\omega_D] \tag{G-18}$$

This is the equivalent of Eq. (E-18) in the two subarray problem. To simplify the result we assume again that the spectrum does not change a great deal over the significant r range. Then

$$\mathrm{Tr}\{(K^{-1} \frac{\partial K}{\partial \Delta_1})^2\} = \frac{2T^2}{\pi^2} (\sum_{r=1}^{\infty} \frac{1}{r^2}) \sum_n \frac{S^2(\omega_n-\omega_D)}{N_0[3S(\omega_n-\omega_D)+N_0]} \tag{G-19}$$

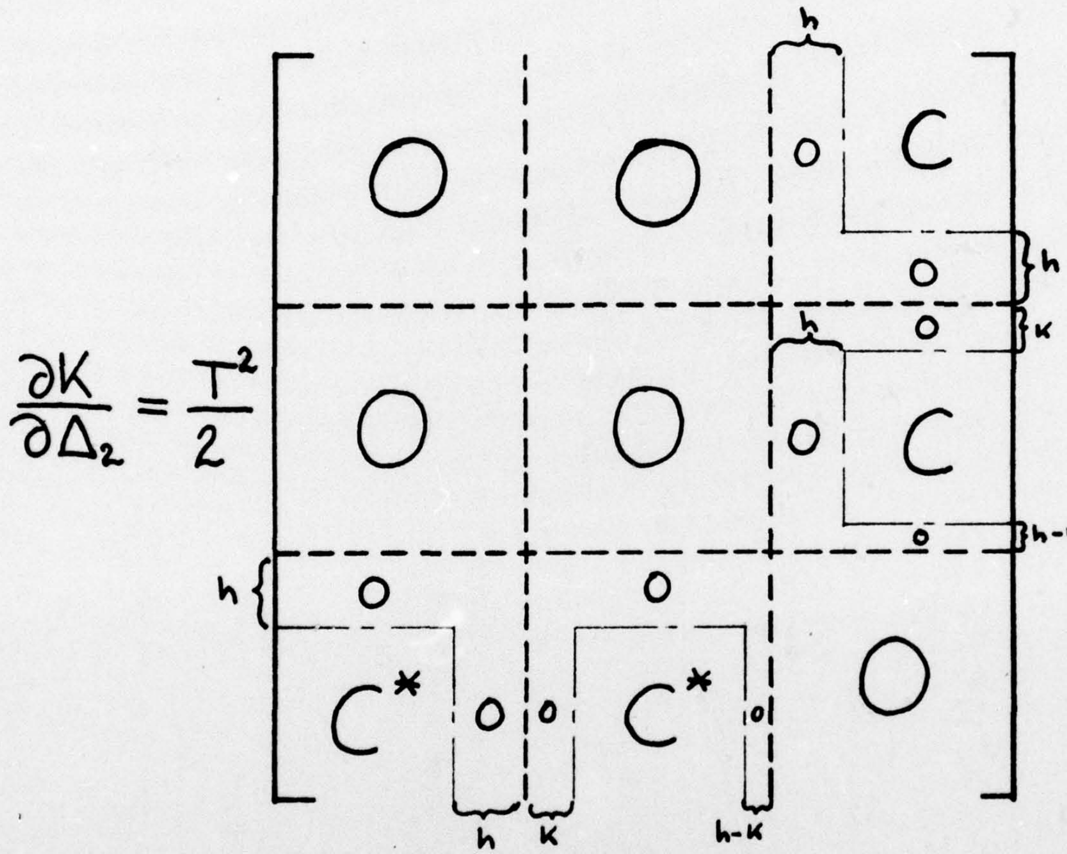
Finally, converting the n sum into an integral and inserting the numerical value $\pi^2/6$ for the r sum

$$\mathrm{Tr}\{(K^{-1} \frac{\partial K}{\partial \Delta_1})^2\} = \frac{T^3}{6\pi} \int_0^{\infty} \frac{S^2(\omega-\omega_D)}{N_0[3S(\omega-\omega_D)+N_0]} d\omega \tag{G-20}$$

The reciprocal of this quantity is the mean square error given by Eq. (139).

Appendix H. Computation of $\mathrm{Tr}(K^{-1} \frac{\partial K}{\partial \Delta_1} K^{-1} \frac{\partial K}{\partial \Delta_2})$

We begin by writing $\frac{\partial K}{\partial \Delta_2}$ in the form of Eq. (G-8). From Eqs. (141)-(143)



(H-1)

The elements of the (N-h) X (N-h) matrix C are given by

$$c_{nm} = \begin{cases} 0 & m = n \\ -\frac{(-1)^{m-n}}{m-n} S[\frac{\pi}{T}(m+n)-\omega_D] & m \neq n \end{cases} \qquad \text{(H-2)}^1$$

---

[1] In terms of the nomenclature of Appendix G, the C of (H-1) is $C^{(h)}$ of G-13.

The matrix product $K^{-1} \frac{\partial K}{\partial \Delta_2}$ is obtained from Eqs. (G-12) and (H-1)

$$K^{-1} \frac{\partial K}{\partial \Delta_2} = \frac{T}{2\pi} \begin{array}{c} \begin{array}{ccccccc} \quad & h & k & \quad & h-k & h & \quad \end{array} \\ \left[\begin{array}{ccccccc} -B^{(h)}C^{*} & 0 & 0 & -B^{(h)}C^{*} & 0 & 0 & A^{(h)}C - B^{(h)}C \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -B^{(h)}C^{*} & 0 & 0 & -B^{(h)}C^{*} & 0 & 0 & -B^{(h)}C + A^{(h)}C \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A^{(h)}C^{*} & 0 & 0 & A^{(h)}C^{*} & 0 & 0 & -B^{(h)}C - B^{(h)}C \end{array}\right] \end{array} \begin{array}{l} \\ \\ \} h \\ \} k \\ \\ \} h-k \\ \} h \\ \\ \end{array} \qquad \text{(H-3)}$$

From Eqs. (G-13) and (H-3)

$$\begin{aligned} \mathrm{Tr}\left(K^{-1} \frac{\partial K}{\partial \Delta_1} K^{-1} \frac{\partial K}{\partial \Delta_2}\right) = \frac{T^2}{4\pi^2} \sum_n \sum_m \{ & B_n B_m c_{mn} c_{nm} - A_n B_m c_{nm} c_{nm} + B_n B_m d_{mn} c_{nm} - B_n A_m d_{nm} c_{nm} \\ & - A_n B_m c_{mn} c_{nm} + B_n B_m c_{nm} c_{nm} + B_n B_m d_{mn} c_{nm} + A_n A_m d_{nm} c_{nm} - B_n A_m c_{mn} c_{mn} \\ & + B_n B_m c_{mn} c_{mn} + B_n B_m c_{nm} c_{mn} - B_n A_m c_{nm} c_{mn} - A_n B_m d_{mn} c_{mn} + A_n A_m d_{mn} c_{mn} \\ & + 2B_n B_m d_{nm} c_{mn} \} \end{aligned} \qquad \text{(H-4)}$$

From Eqs. (G-10) and (H-2) it is apparent that

$$c_{nm} = -c_{mn} = d_{mn} = -d_{nm} \tag{H-5}$$

Hence Eq. (H-4) becomes (after a few steps of algebra)

$$\mathrm{Tr}(K^{-1} \frac{\partial K}{\partial \Delta_1} K^{-1} \frac{\partial K}{\partial \Delta_2}) = \frac{T^2}{4\pi^2} \sum_n \sum_m (4B_n B_m + A_n B_m + B_n A_m - 2A_n A_m) c_{nm}^2 \tag{H-6}$$

Now substituting the definitions (G-6), (G-7) and (H-2) and making the change of variables $m = n+r$

$$\mathrm{Tr}(K^{-1} \frac{\partial K}{\partial \Delta_1} K^{-1} \frac{\partial K}{\partial \Delta_2}) =$$

$$-\frac{T^2}{4\pi^2} \sum_{r \neq 0} \frac{1}{r^2} \sum_n \frac{3[S(\omega_n - \omega_D) + S(\omega_{n+r} - \omega_D)] + 2N_0}{N_0[3S(\omega_n - \omega_D) + N_0][3S(\omega_{n+r} - \omega_D) + N_0]} S^2[\frac{\pi}{T}(2n+r) - \omega_D] \tag{H-7}$$

This expression is exactly half of $\mathrm{Tr}\{K^{-1} \frac{\partial K}{\partial \Delta_1})^2\}$ as given by Eq. (G-18). Since

$$\mathrm{Tr}\{(K^{-1} \frac{\partial K}{\partial \Delta_2})^2\} = \mathrm{Tr}\{(K^{-1} \frac{\partial K}{\partial \Delta_1})^2\} \tag{H-8}$$

The coupling coefficient ρ defined by Eq. (140) assumes the numerical value

$$\rho = \frac{1}{4} \tag{H-9}$$

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